The Journey Toward General Relativity

Part 1: 1907–1912

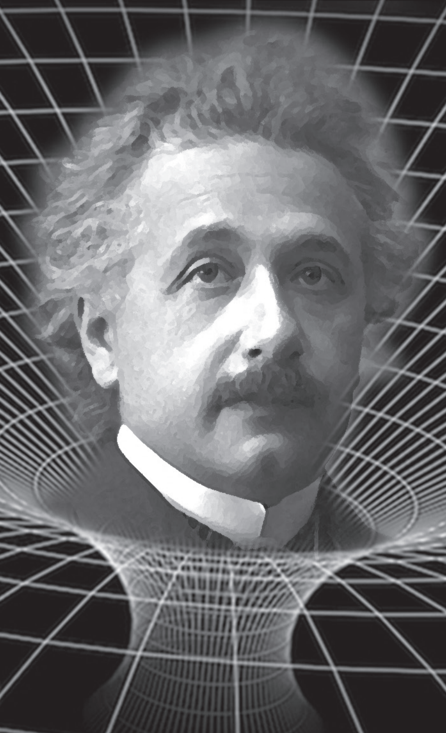

Image by Matt Payne

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Just over one century ago, on November 25, 1915, Albert Einstein presented to the Prussian Academy of Sciences the completed version of his general theory of relativity. His journey toward that theory had begun in 1907, was interrupted until 1911, and then continued with nonstop intensity until the end of 1915.[1]

Despite the great triumphs of the 1905 papers that introduced special relativity through electrodynamics,[2,3] they could not be the last word, because they excluded accelerated reference frames and gravitation. Those gaps weighed heavily on Einstein's mind after 1905. In 1907 Einstein was asked to write a review of special relativity for the prestigious journal *Jahrbuch der Radioaktivität und Elektronik*.[4] He took this opportunity to begin extending relativity to gravitation. This effort appeared in the last part of the article, Sec. V, which the editor received on December 4, 1907. Einstein's close friend and biographer Abraham Pais notes, "It is here that he begins the long road from the special theory to the general theory of relativity."[5]

The "happiest thought"

In 1920, Einstein recalled that his attempts between 1905 and December 1907 to extend relativity to gravitation "did not satisfy me because they were based on physically unfounded hypotheses….Then there occurred to me the happiest thought of my life…"[6] That thought was reminiscent of the electrodynamics puzzle with which Einstein introduced one of the 1905 relativity papers:[2] when a magnet passes through a coil of conducting wire, the changing magnetic flux as observed in the coil's reference frame induces an *electric* force on charges in the coil. But an observer aboard the magnet sees the coil's charges carried with nonzero velocity past the magnet, experiencing a *magnetic* force. The electric field can be transformed away by a change of reference frame. Einstein's "happiest thought" was his realization that a gravitational field could also be transformed away, at least locally, by a change of reference frame. In the 1920 manuscript he wrote, "The gravitational field has only a relative existence in a way similar to the electric field generated by magnetoelectric induction. *Because for an observer falling freely from the roof of a house there exists—at least in his immediate surroundings—no gravitational field*"(Einstein's emphasis).[6] In a lecture given at Kyoto University in Japan in December 1922 he again recalled,[7] "The breakthrough came suddenly one day. I was sitting on a chair in my patent office in Bern. Suddenly a thought struck me: If a man falls freely, he would not feel his weight. I was taken aback. This simple thought experiment made a deep impression on me. This led me to the theory of gravity."

Since gravity could be transformed away locally by going into a free-fall frame, Einstein saw that the principle of relativity—the postulate that laws of physics must not depend on the frame of reference—when extended to include *accelerated* frames, could be a theory of gravitation. The 1907 *Jahrbuch* paper was his first public attempt to include gravity within the principle of relativity.

Uniform acceleration and the relativity of time

Let us follow Einstein's 1907 treatment of the relativity of time and length for a reference frame moving with constant acceleration relative to inertial ones. Einstein had at his service the tools of the equivalence principle, the principle of relativity, and the results of his 1905 papers. He began with kinematics, and with clever insight applied his tools to *three* reference frames. Two of them were inertial, and the third one was accelerated. He called them S, S', and Σ, respectively; here I call them the lab frame L with coordinates (t,x), the coasting rocket frame CR with coordinates $(t'x')$, and the accelerated rocket frame AR with coordinates (t'',x'').[8] The layout is the usual one found in introductory special relativity discussions. The CR frame moves with constant velocity v relative to L; the x and x' axes are parallel and CR moves in the direction of $+x$; and when their origins coincide, the arrays of synchronized clocks in both frames read $t = 0$ and $t' = 0$.[9] Thus transformations from (t,x) to (t',x') coordinates are described by the familiar form of the Lorentz transformation:

$$t' = \gamma(t - vx/c^2), \qquad (1)$$
$$x' = \gamma(x - vt), \qquad (2)$$

where $\gamma \equiv (1 - v^2/c^2)^{-1/2}$, and c denotes the speed of light in vacuum. With Einstein, we further suppose the AR frame moves with acceleration a relative to L, with its x'' axis parallel to those of x and x', and $t'' = 0$ when the three origins coincide, at which event the AR begins accelerating from rest relative to L. Hence, relative to the lab frame, the location of CR and AR origins are, respectively, $x_{CR} = vt$ and $x_{AR} = \frac{1}{2}at^2$, and the velocity of each rocket relative to L is $v_{CR} = v$ and $v_{AR} = at$. To first order in v/c (an approximation used throughout Sec. V), AR accelerates at the same rate a relative to both L and CR.[10]

Designate as E(v) the event were AR has the same velocity as the CR moving at speed v relative to L. At event E(v), CR and AR are instantaneously at rest relative to one another. Einstein asks about the kinematics of AR, relative to L and CR, for a short time interval after E(v). If the acceleration a and the time interval following E(v) are sufficiently small, then the relation between AR and CR is given *approximately* by the Lorentz transformation. Further approximations result by dropping terms quadratic in v/c and a. In 1907 Einstein was willing to sacrifice rigor to get a feel for the problem—always a good starting strategy.

With Einstein, we seek the effect of acceleration on coordinate transformations between inertial and the AR frames. At event E(v), by Eq. (1), for which $x = vt$, clocks aboard the CR frame read time t', where [11]

$$t' = \gamma t\,(1 - v^2/c^2) \approx \mathrm{t}. \qquad (3)$$

Similarly, for any two spatial points 1 and 2 that are the coordinates of events simultaneous in the lab frame, we find by Eq. (2), to first order in v/c, that $x_2' - x_1' \approx x_2 - x_1$. Notice that the time dilation and length contraction effects of special relativity are effectively being ignored; Einstein seeks the effects of gravitation as acceleration on kinematics. But the Lorentz transformation provides the necessary tool, because the second term on the right-hand side of Eq. (1) will turn out to be crucial, where v in Eq. (1) will be replaced with at, as we shall see.

Consider next the time interval between t and $t + \delta$, where t is the lab time coordinate of E(v), and δ is small. With Einstein, we ask about the relativity between L and AR for this interval. At some time t' within this interval [resetting $t' = 0$ in the CR at E(v)], relative to CR the accelerated rocket's origin is located at $x' = \frac{1}{2}at'\,2$ and moves with velocity $v' = at'$. Now transform the CR time coordinate to the AR coordinates, assuming the acceleration to be modest enough that the transformation is approximately Lorentzian within a brief interval after E(v). By Eq. (1), we obtain

$$t'' = \gamma'[t' - (at')(\tfrac{1}{2}at'^2)/c^2] \qquad (4)$$

so that, to order v/c, $t'' \approx t' - O(a^2) \approx t - O(a^2)$, where $O(a^2)$ means terms of order a^2.[12] Also, as it was between L and CR, to first order in v/c, the length contraction reduces to $x_2'' - x_1'' \approx x_2' - x_1'$.

Now comes the main point. Consider two events, E1 and E2. They are simultaneous in the CR frame if and only if $t_1' = t_2'$ exactly. By Eq. (1), in terms of lab coordinates this means

$$t_2 - t_1 = v(x_2 - x_1)/c^2. \qquad (5)$$

Let E1 be E(v), and let E2 be an arbitrary event that occurs shortly after E(v), within the lab time interval t to $t + \delta$. Thus $v = at_1$, and by our previous results we may set $t_2 \approx t_2'' + O(a^2)$ and $x_2 - x_1 \approx x_2'' - x_1''$. Equation (5) can now be expressed in terms of L-to-AR coordinates as

$$t_2'' = t_1\,[1 + at_1\,(x_2'' - x_1'')/c^2] + O(a^2). \qquad (6)$$

Noting that E(v) occurs at the origin of the AR frame, we set $x_1'' = 0$ and drop the subscripts. To first order in a we obtain [13]

$$t'' = t\,(1 + ax''/c^2). \qquad (7)$$

Now comes the punch line: We invoke Einstein's postulate of the local equivalence of acceleration and a gravitational field, and set $a = -g$. In other words, when the AR moves with acceleration a in the $+x$ direction relative to the lab, a passenger aboard the accelerated rocket cannot locally distinguish this acceleration from a gravitational field g in the $-x''$ direction, where $|g| = |a|$. The gravitational field g and its potential Φ are related by

$$\Phi(x'') = -\int g\,dx'' + const., \qquad (8)$$

which yields, for uniform $g = -a$, $\Phi(x'') = ax''$ with the integration constant set to zero. This turns Eq. (7) into

$$t'' = t\,(1 + \Phi/c^2). \qquad (9)$$

In his 1905 relativity papers, Einstein introduced *global* Lorentz invariance, where a unique velocity v describes the relative motion between two inertial frames; the same Lorentz transformation held *everywhere* between those frames. But now, the *local* equivalence between accelerated frames meant that if he held on to the principle of relativity, then *global* Lorentz transformations would have to go; in general, the Lorentz transformation only holds *locally*. Abraham Pais observed that, in 1907, "Others might have shied away from the equivalence principle in order to retain the global invariance. Not so Einstein. With a total lack of fear he starts on the new road. For the next eight years he has no choice. He has to go on."[14] Pais put the 1907 Sec. V into perspective. It "does not have the perfection of the 1905 paper on special relativity. The approximations are clumsy and mask the generality of the conclusions. Einstein was the first to say so, in 1911… Despite all that, I admire this article at least as much as the perfect relativity paper of 1905, not so much for its details as for its courage…."[15]

Despite its shortcomings, Eq. (9) implies predictable consequences. Einstein mentioned them in the *Jahrbuch* paper, and revisited them in more detail when he returned to this subject in 1911. These include gravitational redshift; position dependence of the speed of light in a gravitational field, and thus the deflection of light rays by massive bodies; and the demonstration that $E = mc^2$ applies to both inertial and gravitational mass. Because my space is limited, here I will merge his 1907 and 1911 discussions of these applications.

After the 1907 paper, Einstein said no more in public about gravitation until 1911. In the interim he was focused on quantum theory and radiation. His life was busy in personal ways, too. In July 1909 he could finally resign his post at the patent office to accept his first faculty position that October, as an associate professor of theoretical physics at the University of Zürich. In July 1910 a second son, Eduard, was born to Albert and Mileva Einstein. The family moved in March 1911 when Albert accepted his first full professorship at Karl-Ferdinand University in Prague.

In Prague, Einstein turned his focus back to gravitation with the 1911 publication "On the Influence of Gravitation on the Propagation of Light."[16] He began with a backward glance. "In a memoir published four years ago, I tried to answer the question whether the propagation of light is influenced by gravitation. I return to this theme, because my previous presentation of the subject does not satisfy me, and for a stronger reason, because I now see that one of the most important consequences of my former treatment is capable of being tested experimentally." In 1907 Einstein was thinking of terrestrial experiments to measure the gravitational deflection of a light ray, and he realized this deflection would be too small to detect with such experiments. By 1911 he realized that astronomers looking beyond Earth might be able to test it. He continues, "For it follows from the theory here to be brought forward, that rays of light, passing close to the sun, are deflected by its gravitational field, so that the angular distance between the sun and a fixed star appearing near to it is apparently increased by nearly one second of arc."[17] He also revisited the 1907 inferences on gravitational redshift and whether $E = mc^2$ applies to both gravitational and inertial mass.

Gravitational redshift

In the 1907 paper Einstein made a qualitative observation in a subsection called "Influence of the Gravitational Field Upon Clocks." He wrote, "If at a point P of the gravitational field Φ there is situated a clock which indicates the local time, then according to Eq. (9) its indications are $1 + (\Phi/c^2)$ greater than the time [t], i.e., it runs $1 + (\Phi/c^2)$ faster than in an identically constructed clock situated at the origin of coordinates."[18] In 1911 he turned this qualitative comment into a quantitative prediction and derived the same result in a much simpler way than the path that led to Eq. (9), this time by starting with the relativistic Doppler shift for light. The Doppler shift says that when a source light at rest relative to the CR gets carried with velocity v along the x-axis relative to L, the light's frequency as observed in the lab frame is [19] $f_{\text{moving}} = f_{\text{rest}}\,\gamma\,(1 + v/c)^{-1}$. Einstein applied this to radiation emitted and detected within a uniformly accelerated frame. Let the light be emitted from point P1, to arrive at a detector at point P2 some distance h away in the same system. If the radiation has the frequency f_1 relative to the clock at P1, then upon the radiation's arrival at P2, that detector moves with velocity $v = at = ah/c = \Phi/c$ (neglecting length contraction effects). Therefore, by the Doppler formula just mentioned, to first order in v/c the radiation would have a greater frequency f_2 at P2,

$$f_2 = f_1(1 + ah/c^2) = f_1\,(1 + \Phi/c^2). \qquad (10)$$

To see consistency with the 1907 result, consider a clock of period T_1 at location 1, where the gravitational potential is Φ_1. At location 2 where the potential is Φ_2, an identical clock has period T_2. According to Eq. (9) their periods are related by

$$\begin{aligned} 1 &= [T_2\,(1 + \Phi_2/c^2)]\,[T_1\,(1 + \Phi_1/c^2)]^{-1} \\ &\approx (T_2/T_1)\,[1 + (\Phi_2 - \Phi_1)/c^2], \qquad (11) \end{aligned}$$

where the binomial expansion has been used assuming $\Phi/c^2 \ll 1$. Equations (10) or (11) give the frequency shift,

$$(f_2 - f_1)/f_1 = (\Phi_2 - \Phi_1)/c^2, \qquad (12)$$

the same as Eq. (10).

Testable predictions follow immediately. In a uniform gravitational field $\Phi = gy$, where $y_1 = 0$ is the level of the floor and $y_2 = h$ is the top of a building, the frequency shift is gh/c^2, the basis of the 1960 Pound-Rebka experiment [20] that offered the first precise terrestrial affirmation of gravitational redshift. The atom in the stronger potential emits light of lower frequency (longer wavelength). For the gravitational potential about a star of mass M, $\Phi = -GM/r$. If point 2 denotes the solar surface where $\Phi_2 = -GM/R$ (M is the Sun's mass $= 2\times10^{30}$ kg,

R is the solar radius ≈ 7×10^8 m, and $G = 6.67\times10^{11}$ Nm^2/kg^2 denotes Newton's gravitational constant), and $\Phi_1 \approx 0$ denotes the gravitational potential at Earth's orbit, Eq. (12) gives $f_2/f_1 = f_1(1 - GM/Rc^2) < f_1$, or $\Delta f / f_1 = 2\times10^{-6}$.[21] Even though he derived Eq. (9) for a *uniform* gravitational field, in the solar potential example Einstein boldly assumed the result also holds for an *inhomogeneous* gravitational field.

Does $E = mc^2$ hold for gravitational mass?

In September 1905 Einstein had showed that an object of mass m corresponds to an energy mc^2.[3] He derived this by considering a body that emits light and therefore suffers a change in mass. That paper was concerned with *inertial* mass, as signaled by its title, "Does the Inertia of a Body Depend on Its Energy Content?" Using a result for the relativity of energy derived in the June 1905 paper,[2] he imagined a source in the lab frame emitting light of energy E above the x-axis at angle φ. As seen by the CR frame, the emitted energy is

$$E' = \gamma E\,[1 - (v/c)\cos\varphi]. \qquad (13)$$

Let the AR frame start from rest and accelerate with acceleration a relative to a lab frame that has no gravity. At $t = 0$ when the AR frame begins accelerating, it simultaneously emits a light pulse of energy E_2 from a point at the distance h beyond the AR origin. A detector resides at the origin that will receive this pulse of radiation. As seen from the L frame, because length contraction is assumed negligible, it takes time h/c between the light being emitted and absorbed, and upon absorption the AR frame moves at speed ah/c relative to L. Applying Eq. (13) and noting that $\varphi = 180^o$, Einstein finds, to first order in v/c, that the energy absorbed is $E_1 \approx E_2(1 + ah/c)$. According to the principle of equivalence, $a = g$ in magnitude, and in a uniform field the potential is $\Phi = gh$. Therefore,

$$E_1 = E_2 + E_2\Phi/c^2. \qquad (14)$$

By virtue of $E = mc^2$, upon reception of the signal the mass equivalent has gone from $m_2 = E_2/c^2$ to $m_1 = m_2 + m_2\Phi$. Therefore, $E = mc^2$ holds for both inertial *and* gravitational mass.[22]

On the speed of light in a gravitational field

In the June 1905 paper that introduced special relativity,[2] Einstein predicted from his postulates the Lorentz transformations [23] of the electric field **E** and the magnetic field **B**, for inertial frames in relative motion with constant velocity **v**.[2] In terms of an unprimed "rest" frame and a primed "moving" frame, to first order in v/c the transformations read [24]

$$\mathbf{E}' \approx \mathbf{E} + \mathbf{v}\times\mathbf{B}/c \qquad \mathbf{B}' \approx \mathbf{B} - \mathbf{v}\times\mathbf{E}/c. \quad (15)$$

In the 1907 paper Einstein applied them to the Faraday-Lenz and Ampère-Maxwell laws to study the relativity of the speed of light in accelerated frames. In the same spirit, but following a different (we hope shorter) route, here we apply them to the electromagnetic wave equation and draw the same conclusion as Einstein. In source-free regions, consider an electromagnetic wave traveling in the x''-direction of the AR frame, with an electric field polarized in the y''-direction and the magnetic field in the z''-direction. An observer in the AR frame writes the homogeneous wave equation [25]

$$\partial^2 E''_y/\partial x''^2 - (1/c''^2)\,\partial^2 E''_y/\partial t''^2 = 0. \qquad (16)$$

Let us transform this to the CR frame shortly after event $E(v)$, when the speed v' of the AR relative to the CR is small, and apply Eqs. (15) in the form

$$\mathbf{E}'' \approx \mathbf{E}' + \mathbf{v}'\times\mathbf{B}'/c, \qquad \mathbf{B}'' \approx \mathbf{B}' - \mathbf{v}'\times\mathbf{E}'/c. \qquad (17)$$

Thus for events nearby in space and time to $E(v)$, the fields in the AR will be given by Eq. (15). With v' in the x''-direction, we find that $E_y'' \approx E'_y - v'B'_z/c$. Recalling that $dx'' \approx dx'$ to first order in v'/c, and leaving dt'' alone for the moment, we find

$$\partial^2 E'_y/\partial x'^2 - (1/c^2)\,\partial^2 E'_y/\partial t''^2 - (v'/c)[\partial^2 B'_z/\partial x'^2 - (1/c''^2)\,\partial^2 B'_z/\partial t''^2] \approx 0. \qquad (18)$$

The wave operators for the electric and magnetic field components vanish separately, and in each case, the denominator that is essentially $c^2(\Delta t'')^2$ becomes $c^2(1 + \Phi/c^2)^2(\Delta t')^2$. This says that, in the presence of acceleration, and thus in the presence of gravitation characterized by the potential field $\Phi(\mathbf{r})$, the speed of light in vacuum is the position-dependent quantity,[26]

$$c(\mathbf{r}) = c_o[1 + \Phi(\mathbf{r})/c^2], \qquad (19)$$

where c_o denotes the speed of light in vacuum without gravity.

Analogous to the speed of light being c/n in a medium of refractive index n, the gravitational potential produces, in effect, a variable index of refraction, $n = (1 + \Phi/c^2)^{-1}$. In the *Jahrbuch* paper Einstein qualitatively observed, "It follows from this that the light rays that are not propagated in the [x''] direction are bent by the gravitational field."[27] In the 1911 paper he offered a quantitative estimate of the deflection, beginning with Huygens' principle (Fig. 1a). Consider a plane wave front AA. Suppose the speed of light, due to the gravitational potential of a nearby body, differs at points 1 and 2, which are the distance ℓ apart, with the potential larger at point 1. By Huygens' principle, the wave front after the elapse of time dt is inclined at the angle φ relative to AA, where $d\varphi = (c_2 - c_1)dt/\ell$ or by Eq. (19), in terms of arc length, $ds = \ell d\varphi = -\Delta\Phi dt/c_o = -(\nabla\Phi\cdot d\mathbf{r})dt/c_o$.[28] Thus the deflection per unit path of light ray is $d\alpha \equiv ds/(c_o dt) = -(\nabla\Phi\cdot d\mathbf{r})/c_o^2$. Let a light ray come from infinity and pass the Sun (mass M and radius R, $\Phi = -GM/r$) at grazing incidence. As shown in Fig. 1b, the angle of deflection follows from

$$\alpha = \frac{2}{c_0^2}\int_\infty^R \frac{GM}{r^2}\,dr = \frac{2GM}{c_0^2 R}; \qquad (20)$$

the 2 coming from symmetry about the point of closest approach. Einstein obtained the numerical value of 0.83 seconds of arc (half of what general relativity would predict four years later), and concluded, "It would be a most desirable thing if astronomers would take up the question here raised. For apart from any theory there is the question whether it is possible with the equipment at present available to detect an influence of gravitational fields on the propagation of light."[29]

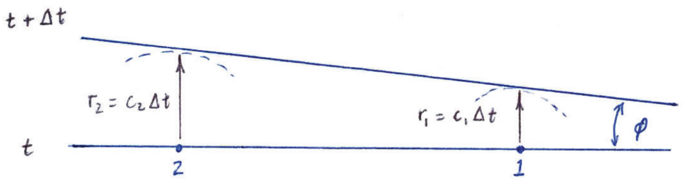

Fig. 1 (a)

Fig. 1 (b)

Fig. 1. (a) The Huygens wave fronts used to estimate light wave deflection. (b) Deflection of a light ray passing the Sun.

An expedition was planned for 1914, but it was cancelled by the outbreak of World War I.[30] While the war was a disaster for humanity, for Einstein's measurement the cancellation turned out to be fortuitous, for although he did not realize it at the time, his 1911 result gave the same deflection as Newtonian gravitational theory. Newton thought of light as a swarm of particles. In *Opticks* (1704) he asked, "Do not bodies act upon light at a distance, and by their action bend its rays…?" Because m cancels out of $\mathbf{F} = m\mathbf{a}$ when $\mathbf{F}$ is the force of gravity, in a uniform field the deflection of a particle of light becomes just another projectile problem, and becomes a scattering problem with the $1/r$ potential.[31] In a little-known paper of 1804, Johann Georg von Soldner calculated the Newtonian deflection and obtained $\alpha = 0.84''$.[32]

Einstein published two more papers on gravitation-as-relativity in February and March 1912.[33] They mark Einstein's first attempt to go beyond gravity as kinematics, into a dynamical field theory by trying out the field $c(\mathbf{r}) \sim \Phi(\mathbf{r})$ of Eq. (14). The 1912 scalar theory did not survive into the final results of 1915, but the issues it raised helped prepare the way. Let us say a few words about the February and March 1912 papers.[34] The departure from Newtonian theory entered with the crucial result, carried over from the 1907 and 1911 papers, that the speed of light is a *scalar field*, where

$$c(\mathbf{r}) = c_o + \Phi(\mathbf{r})/c_o. \qquad (21)$$

In Newtonian gravitation theory—a *static* field theory—the potential follows from a mass distribution according to Poisson's equation, $\nabla^2\Phi = 4\pi G\rho$, with ρ the mass density. Introducing $\sigma_m = \rho c_o^{\,2}$ as the energy density of matter, Einstein's *ansatz* turns Poisson's equation into

$$\nabla^2 c = 4\pi G\, c_o^{-3}\sigma_m, \qquad (22)$$

still a static field theory. However, a particle of mass m moving *through* the static field would, in Einstein's 1912 theory, have the equation of motion similar to that of special relativity, even when the speed of light is a function of position. In special relativity, a free particle moves such that the integral of proper time is stationary, $\delta\!\int\! d\tau = 0$, where $c^2d\tau^2 = c^2dt^2 - d\mathbf{r}\cdot d\mathbf{r}$. Allowing c to be a function of spatial coordinates, this gives [35]

$$d/dt(\gamma\mathbf{v}/c) = -\gamma\nabla c, \qquad (23)$$

where $\mathbf{v} = d\mathbf{r}/dt$ (AR frame coordinates, primes dropped) and $\gamma = (1 - v^2/c^2)^{-1/2}$. Pais writes, "Einstein was stirred by the fact that [the equations of motion] still apply if c is a static field!...It is hard to doubt that this insight guided Einstein to the ultimate form of the mechanical equations of general relativity, in which eq. [$\delta\!\int\! d\tau = 0$] survives, while [the expression for $d\tau^2$] is generalized further."[36]

Pais wonders how Einstein could consider a *static* gravitational field coupled to a *dynamic* electromagnetic field. Pais never had the opportunity to ask Einstein about this, because he never saw these 1912 papers until after Einstein had passed away. So Pais imagined that Einstein pushed as far as he could with the simplest assumptions possible, until the inevitable contradictions would suggest to him the next step.[37] Such an instance came up at once. Since electrodynamics and gravity are coupled through Eq. (22), Einstein knew that the mass density of Poisson's equation for Φ would have to be generalized beyond the Newtonian paradigm to include the energy density of the electromagnetic field. Letting σ_{me} denote the sum of σ_m and the electromagnetic energy density $\sigma_e = \tfrac{1}{2}\varepsilon_o E^2 + B^2/2\mu_o$, the resulting Poisson equation, $\nabla^2 c = 4\pi G\, c_o^{-3}\sigma_{me}$, unfortunately did not satisfy local energy and momentum conservation.[38] The static Newtonian gravitational field $\mathbf{g}$ carries the energy density $\sigma_g = -g^2/8\pi G = -(\nabla\Phi)^2/8\pi G = -c_o(\nabla c)^2/8\pi G$ and thus Einstein postulated that

$$\nabla^2 c = 4\pi G\, c_o^{-3}(\sigma_{me} + \sigma_g), \qquad (24)$$

or in terms of Φ,

$$\nabla^2\Phi + \tfrac{1}{2}\, c_o^{-2}(\nabla\Phi)^2 = 4\pi G\,\sigma_{me}, \qquad (25)$$

a manifestly nonlinear theory. Pais quotes Einstein: "It has been a grave decision to make this last modification of the c-field equation, Einstein wrote, 'since [as a result] I depart from the foundation of the unconditional equivalence principle.'" In the absence of ponderable matter or electromagnetic radiation, so that $\sigma_{me} = 0$, Eq. (22) becomes $\nabla^2 c = 0$ in the AR frame, which by the unconditional equivalence principle would have to produce $\nabla^2 c = 0$ in the lab frame too. But if $\sigma_{me} = 0$ in Eq. (25), then $\nabla^2 c \neq 0$, *unless* the region where the equation is applied is so small that the gradient of the potential is negligible. Einstein continued, "It seems that [the equivalence principle] holds only for infinitely small fields…" Pais comments, "This is the dawn of the correct formulation of equivalence as a principle that holds only locally."[39]

With the February and March 1912 papers that Einstein wrote while living in Prague, he saw far enough into gravitation with relativity to see that if the principle of relativity was universal, then the principle of equivalence and Lorentz invariance could hold only locally, equations of motion could nevertheless follow from a variational principle, and gravity would couple to itself and thus its field equations be nonlinear. But he still lacked the language to pull it all together.

In August 1912, Einstein and his family moved back to Zürich. In the meantime he decided that the scalar theory would not do the job and space, as well as time, had to be non-Newtonian. For instance, in the early 1912 papers he mused on the possibility that the ratio of a circle's circumference to its diameter might not be π, because of a length contraction when the circle spins about its axis. In his Kyoto speech 10 years later he recalled, "If all accelerated systems are equivalent, then Euclidean geometry cannot hold in all of them…This problem remained insoluble to me until 1912, when I suddenly realized that Gauss's theory of surfaces holds the key for unlocking this mystery. I realized that Gauss's [non-Euclidean] surface coordinates had a profound significance. However, I did not know at that time that Riemann had studied the foundations of geometry in an even more profound way. I suddenly remembered that Gauss's theory was contained in the geometry course given by Geiser when I was a student…I realized that the foundations of geometry have physical significance. My dear friend the mathematician Grossman was there when I returned from Prague to Zürich. From him I learned for the first time about Ricci and later about Riemann…"[7, 40]

Marcel Grossman and Albert Einstein were students together at the Zürich Polytechnic. Grossman was a mathematician who knew tensor calculus. When Einstein returned to Zürich to teach at his alma mater (renamed the ETH in 1911), the former classmates got together in a collaboration that was to prove fruitful indeed, as we shall see in the next installment.

References

[1] Much of the history related here is adapted from *Subtle is the Lord: The Science and the Life of Albert Einstein* by Abraham Pais (Oxford University Press, 1982). Like other biographers, Pais had access to primary documents. Unlike most Einstein biographers, Pais knew Einstein personally and the physics thoroughly.
[2] Albert Einstein, "On the Electrodynamics of Moving Bodies," *Ann. der Phys.* **17** (1905) 891–921; see John Stachel (ed. and trans.) and Roger Penrose, *Einstein's Miraculous Year: Five Papers that Changed the Face of Physics* (Princeton University Press, 1998); or H.A. Lorentz, A. Einstein, H. Weyl, and H. Minkowski, *The Principle of Relativity*,W. Perrett and G.B. Jeffery, tr. (Methuen, 1923; reprinted by Dover, 1952). In addition, Einstein's 1905 relativity papers are annotated and expressed in modern notation in the "Elegant Connections in Physics" column: "On the Electrodynamics of Moving Bodies (Part A: Kinematics) by Albert Einstein," *SPS Observer* (Fall 2005), 10–15 and "On the Electrodynamics of Moving Bodies (Part B: Electrodynamics) and Its Corollary, $E = mc^2$, by Albert Einstein," *SPS Observer* (Winter 2005), 10–20.
[3] Albert Einstein, "Does the Inertia of a Body Depend on Its Energy Content?", *Ann. der Phys.* **18** (1905) 639–641; in ref. 2 see Stachel & Penrose, or *The Meaning of Relativity*, or Part B of "Elegant Connections."
[4] Albert Einstein, "On the Principle of Relativity and the Conclusions Drawn Therefrom," *Jahrbuch für Radioaktivität und Elektronik* **4** (1907) 411. For an annotated translation, see the three-part series by H.M. Schwartz: "Einstein's Comprehensive 1907 Essay on Relativity, Part I," *Am. J. Phys.* **45** (6), June 1977, 512–517; "Einstein's Comprehensive Essay on Relativity, Part II," *Am. J. Phys.* **45** (9), September 1977, 811–817; "Einstein's Comprehensive 1907 essay on Relativity, Part III," *Am. J. Phys.* **45** (10), October 1977, 899–902. The section of Einstein's 1907 paper that extends relativity to a frame moving with constant acceleration relative to inertial ones occurs in his Sec. V, whose translation appears in part III of Schwartz's paper. See also Pais, ref. 1,178–183.
[5] Pais, ref. 1, 178.
[6] The 1920 manuscript, never published, resides in the Pierpont Morgan Library, New York, NY; see Pais, ref. 1, 177–178.
[7] Albert Einstein, "How I Created the Theory of Relativity," translated by Yoshimasa A. Ono, *Physics Today*, Aug. 1982, 45–47.
[8] I use the "rocket frame" motif along the lines of *Spacetime Physics* by John A. Wheeler and Edwin Taylor (Freeman, 1966).
[9] The y and z dimensions are suppressed because we consider here no relative motion in those directions. Einstein considered them, which meant considering the *shape* of an accelerated body. He concluded "we do not have therefore to assume any influence of the acceleration on the shape of the body" (see Schwartz, ref. 4).
[10] Consider a particle moving with velocity v relative to the L frame and v' relative to the CR frame, where v_o denotes the constant relative velocity between the two inertial frames, so that, by the Lorentz transformation, $v' = (v - v_o)/(1 - vv_o/c^2)$. The particle's acceleration relative to L is $a = dv/dt$, and $a' = dv'/dt'$ relative to CR. It follows that $a' = (dv'/dt)(dt/dt') = a\gamma_o^{-3}(1 - vv_o/c^2)^{-3}$ where $\gamma_o = (1 - v_o^2/c^2)^{-1/2}$. Thus, $a' \approx a$ to order v/c.
[11] Quantities that will be approximated as unity (such as γ) or zero (such as terms of order v^2/c^2) are initially retained so the logic of the steps will be easier to follow.
[12] If the $O(a^2)$ terms were neglected from the outset, these results could have been written at once simply by setting $\gamma \approx 1$ for the time dilation and length contraction formulas. But determining the effect of acceleration on the kinematics was the point of the calculation. Albert Einstein was not one to take short cuts.
[13] Einstein speaks of σ as "local time" and τ as "time at the origin"; our Eq. (9) is his *Jahrbuch* Eq. (30a), which reads $\sigma = \tau[1 + (\Phi/c2)]$. He comments, "We have defined two kinds of time for [AR]. Which of the two definitions do we have to utilize in the different cases?...For the definition of physical quantities at a given place of the gravitational field, we quite naturally utilize the time σ….But if one deals with a phenomenon that necessitates the simultaneous consideration of objects situated at places of different gravitational potential, then we must employ the time σ." One can interpret our t'' (or Einstein's σ) as proper time for the situation under consideration, and our Eq. (9) [his Eq. (30a)] as the low-speed, gravitational field contribution to time dilation. Proper time is the time as recorded on the wristwatch of a passenger in AR, viz., a passenger in a gravitational field, and t is the time "at the origin" (Einstein's words) as recorded by a clock back at the origin of the lab frame (see E. Taylor and J.A. Wheeler's discussion of local time vs. "far-away time" in the Schwarzschild metric, *Exploring Black Holes: An Introduction to General Relativity* (Addison, Wesley, Longman, 2000), chs. 1, 2. In addition, from the Schwarzschild solution of general relativity, the relation between proper time dt_p and the time as recorded by an observer at some far-away station that receives data from local clocks around a star of mass M, for which $\Phi = -GM/r$, the relation between proper and far-away time is $dt_p = (1 - 2MG/rc^2)1/2dt \approx (1 + \Phi/c2)dt$, which agrees with our interpretation of Einstein's 1907 result.
[14] Pais, ref. 1, p. 183.
[15] *ibid*., 182–183.
[16] Albert Einstein, "On the Influence of Gravitation on the

Propagation of Light," *Ann. der Phys.* **35** (1911) 898, or *The Principle of Relativity* (ref. 2), 97–108.
[17] *The Principle of Relativity*, ref. 16, 99.
[18] See Schwartz, ref. 4, paper III, 900–901.
[19] E.g., B.W. Carroll and D.A. Ostlie, *An Introduction to Modern Astrophysics* (Addison-Wesley, 1996), 108.
[20] R.V. Pound and G.A. Rebka, "Apparent Weight of Photons," *Phys. Rev. Lett.* **4** (7), (1960), 337–401. These authors cited Einstein's 1911 paper, ref. 16. See also Taylor and Wheeler, ref. 8, 154–155.
[21] Einstein, ref. 16 (Perret & Jeffrey, tr.), 108.
[22] Einstein, ref. 4 (or Schwartz III, 902) and Einstein, ref. 16 (or Perret & Jeffrey, tr., 101–102).
[23] Introduced by H.A. Lorentz in "Electromagnetic Phenomena in a System Moving With Any Velocity Less Than That of Light," *The Principle of Relativity*, ref. 2, 9–34 ([original, Proceedings of the Academy of Sciences of Amsterdam, **6** (1904)].
[24] E.g., J.D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, 1975), 552. With no approximation the transformations read $\mathbf{E}' = \gamma(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}) - \gamma^2(\gamma+1)^{-1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E})$ and $\mathbf{B}' = \gamma(\mathbf{B} - \boldsymbol{\beta}\times\mathbf{E}) - \gamma^2(\gamma+1)^{-1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B})$, where $\boldsymbol{\beta} = \mathbf{v}/c$.
[25] Here I follow the same *strategy* as Einstein used on Faraday-Lenz and Ampère-Maxwell laws.
[26] Referring back to note [13], in the 1907 paper Einstein uses $\partial/\partial\tau = (1 + \Phi/c^2)\,\partial/\partial\sigma$, equivalent to our method here.
[27] Einstein, ref. 4 (Schwartz III, 902), Einstein, ref. 16 (Perret & Jeffrey) 107. In the term Φ/c^2, one may use either $c(\mathbf{r})$ or c_o, since they are the same to first order in Φ/c^2. Einstein used c, I use c_o for definiteness.
[28] The minus sign appears because Φ gets smaller in magnitude as c gets larger, and the gradient points in the direction of increasing Φ.
[29] Einstein, ref. 16, *The Principle of Relativity* (ref. 16), 108.
[30] W. Isaacson, *Einstein: His Life and Universe* (Simon & Schuster, 2007), 202–205.
[31] "Elegant Connections in Physics: Starlight Deflection in Newtonian Mechanics," *SPS Observer*, www.spsnational.org/the-sps-observer/physics-connections.
[32] J.G. von Soldner, *Berliner Ast. Jahrbuch* (1804), 161; see also Pais, ref. 1, 194.
[33] A. Einstein, "Lichtgeschwindigkeit und Statik des Gravitationsfeldes," *Annalen der Physik* **38** (1912) 355–369; "Zur Theorie des Statischen Gravitationsfeldes," *Ann. der Phys.* **38** (1912) 443–458.
[34] See Pais, ref. 1, 201–206.
[35] $\delta\int d\tau = 0$ is a problem in the calculus of variations. Let us illustrate with one spatial dimension. Write $d\tau = [c^2 - (dx/dt)^2]^{1/2}dt$, giving the Lagrangian $L = [c^2 - v^2]^{1/2}$ so that the Euler-Lagrange equation, $\partial L/\partial x = d/dt(\partial L/\partial v)$, gives $-(dc/dx) = d/dt(\gamma v/c)$. Generalizing to three spatial dimensions yields Eq. (23).
[36] Pais, ref. 1, 203.
[37] *Ibid*., 204.
[38] The electromagnetic field carries energy density σ_e and momentum density S/c^2 where $\mathbf{S} = (\mathbf{E}\times\mathbf{B})/\mu_o$ is Poynting's vector [e.g., see David Griffiths, *Introduction to Electrodynamics*, 2nd ed. (Prentice-Hall, 1999), ch. 8]. If the charged particles are also coupled to gravity, energy and momentum conservation are not accounted for only by the electromagnetic field and its coupling to matter; gravitational energy density and its current must be included.
[39] Pais, ref. 1, 205.
[40] Pais, ref. 1, 211–212, offers minor differences in translation as ref. 7; I have used the Pais translation here.