This is the second part in a series outlining Albert Einstein’s development of the general theory of relativity. In Part 1 we saw that by mid-1911 Einstein¹

- knew that, in general, the equivalence principle held only locally, and likewise the Lorentz transformation, implying the need of a larger invariance group (thus, physical laws more complex than Maxwell’s equations for electrodynamics were needed for gravitation);
- had confidence in an action principle for a particle falling freely in a gravitational field;
- realized the gravitational field equations must be nonlinear because gravitational field energy is a source of gravitation;
- had calculated gravitational redshift and predicted 0.83” deflection of a light ray grazing the Sun, essentially the same as the Newtonian prediction.

He had come this far with a scalar field theory that treated the speed of light $c$ as a function of spacetime coordinates $x^\mu$, $c = c(x^\mu)$. In this notation, $x^\mu$ indicates the $\mu$ component of the position vector.

In August 1912 Einstein and his family moved from Prague back to Zürich. Seven years earlier in 1905, he had completed his PhD from the University of Zürich, and in 1900 he had earned an undergraduate diploma from ETH, the Swiss Federal Institute of Technology,² where Marcel Grossmann (Fig. 1) was a friend and classmate. In early 1912 Grossmann was a professor of mathematics and dean at ETH, and sounded out Einstein regarding returning there as a faculty member.

About the time his family returned to Zürich, it abruptly became clear to Einstein that his scalar $c$-field theory of gravitation would not be sufficient, and non-Euclidean geometry was “the correct mathematical tool” for what would become the general theory of relativity. This was apparent, he later wrote, “because of the Lorentz contraction in a reference frame that rotates relative to an inertial frame, the laws that govern rigid bodies do not correspond to the rules of Euclidean geometry.” As seen from an inertial frame, the ratio of circumference to diameter for a spinning disc is not equal to $\pi$. In 1912 Einstein “suddenly realized that Gauss’s theory of surfaces holds the key… I suddenly remembered that Gauss’s theory was contained in the geometry course given by [Carl Friedrich] Gauss when I was a student.”³
Back in 1827, Carl Friedrich Gauss published *Disquisitiones generales circa superficies curvas* (General Investigation of Curved Surfaces), where he introduced the notion of studying geometry through its “inner” properties accessible to a geometer living in the space, without relying on a higher-dimensional embedding space. For instance, a fundamental inner property is the distance $ds$ between infinitesimally nearby points, where $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ (noting that repeated indices are summed over, which is also called the Einstein summation notation), with $dx^\mu$ a coordinate differential. As an example, in two dimensions the distance between two points that are separated by $dx$ in the $x$ direction and $dy$ in the $y$ direction would be $ds^2 = 1dx^2 + 1dy^2$.

Conceptually, we know that the distance between two points on a sheet of paper is the same whether it lies flat or gets rolled into a cylinder; the third dimension need not be used to determine distance on the paper. In contrast, the Pythagorean theorem gives the distance only between infinitesimally nearby points on the surface of a sphere. Thus a college campus can be mapped with a Euclidean coordinate system, but mapping the Earth’s entire surface on a single such grid produces distortion, because a spherical surface and a plane have different inner properties. For two-dimensional spaces, Gauss derived a complicated expression for the “Gaussian curvature” $K$ that produces from the metric tensor components $g_{\mu\nu}$ and their derivatives a number related to the curvature of the space. For the Euclidean plane example, the diagonal terms $g_{11} = g_{22} = 1$ while the off-diagonal terms $g_{12} = g_{21} = 0$, and Gauss’s expression gives $K = 0$. But for the two-dimensional surface of a sphere of radius $R$, the Gaussian curvature $K = 1/R^2$ everywhere. Such ideas became applicable to physics when, in 1908, Hermann Minkowski rewrote special relativity as geometry by showing that the distance $ds$, or rather, its square between two events in the spacetime, may be expressed

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \equiv \eta_{\mu\nu}dx^\mu dx^\nu, \quad (1)$$

which has zero curvature, a so-called “flat” spacetime.

A spinning disc is an accelerated disc, and by the principle of equivalence, an accelerated frame is locally indistinguishable from a gravitational field. If the spacetime metric can be changed by acceleration, it could also be changed by gravitation. Therefore, in the presence of a gravitational source, the metric tensor components would change from the $\eta_{\mu\nu}$ of the Minkowskian spacetime of Eq. (1) to some more general $g_{\mu\nu}$. Thus gravitation can be visualized as the curvature of spacetime!

Einstein set himself to the task of solving this problem of finding the $g_{\mu\nu}$ supposing the gravitational source to be given. When recalling the work of 1912, he later said:5

*I had the decisive idea of the analogy between the mathematical problem of general relativity and the Gaussian theory of surfaces only in 1912... without being aware at that time of the work of [Georg] Riemann, [Gregorio] Ricci, and [Tullio] Levi-Civita. This [work] was first brought to my attention by my friend Grossmann when I posed to him the problem of looking for generally covariant tensors whose components depend only on derivatives of the [metric tensor] coefficients...*

Einstein speaks here of tensor calculus applied to non-Euclidean geometries in multidimensional spaces. Riemann, Ricci, and Levi-Civita occupy honored places in the pantheon of distinguished mathematicians who, in the nineteenth and early twentieth centuries, generalized the work of Gauss and other non-Euclidean geometry pioneers.

On October 29, 1912, Einstein wrote these now-famous lines in a letter to Arnold Sommerfeld:6

*I am now occupied exclusively with the gravitational problem, and believe that I can overcome all difficulties with the help of a local mathematician friend. But one thing is certain, never before in my life have I troubled myself over anything so much, and that I have gained great respect for mathematics, whose more subtle parts I considered until now, in my ignorance, as pure luxury! Compared with this problem, the original theory of relativity is childish.*

### Interlude: Digression on Tensors

At this point let us pause to offer a few informal notes for readers who, like Einstein in 1912, might appreciate a quick brushup on tensors. Here we seek descriptions, not definitions; strategic ideas, not tactical details that can be found in textbooks.7

In the jargon of tensors, a scalar is a tensor of rank (or order) zero, and a vector a tensor of rank 1. Accordingly, vector components carry one index; scalars have no indices. Higher-rank examples include the two-index rank-2 metric, the quadrupole, and inertia tensors.
Tensors are formally defined by how they transform under a change of coordinates $x^i \to x'^i$. For $\lambda$ to be a scalar, $\lambda' = \lambda$ for all coordinate transformations. Since vectors are displacements, then a vector component transforms the same as a coordinate displacement. In other words, under a coordinate transformation, since each new coordinate is a function of all the old coordinates, $x^i = x^i(x')$, by the chain rule it follows that $dx^i = (\partial x^i/\partial x^i)dx^i' = A^i_\mu dx^\mu$, where the $A^i_\mu$ are the coefficients parameterizing the transformation. A vector with components $A^i$ transforms in the same way: $A'^i = A^i_\mu A'^\mu$. Informally, a rank-2 tensor is merely a product of two-vector, or rank-1 tensor components, $T'^\mu = A^i_\mu A'^i$, and thus in terms of the formal definition, transforms as $T'^{\mu\nu} = A^i_\mu A'^i_\nu T^{\mu\nu}$—and so on for tensors of higher rank.

Writing a scalar product as a sum of products of vector components is maintained by introducing, corresponding to $dx^\mu$, its dual $dx^\nu$ according to $dx^\nu = g^\nu_\mu dx^\mu$. For instance, in special relativity with $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z)$ and $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, we find that $dx^0 = dt$ but $dx^1 = -dx$.

Coordinate systems are not part of nature but are mappings introduced for convenience. Therefore physics equations must be written in languages that transcend the choice of coordinates. This can readily be done by writing them with tensors. For if the equation $T'^\mu = S'^\mu$ holds in one reference frame, and if these terms are tensor components, then under a coordinate transformation the relation $T'^{\mu\nu} = S'^{\mu\nu}$ also holds—as go the coordinates, so go tensor components. A scalar is invariant under the transformation $(\lambda' = \lambda)$, but vectors and higher-rank tensors transform covariantly—although $T'^\mu = S'^\mu$ and $T'^{\mu\nu} = S'^{\mu\nu}$, $T^{\mu\nu}$ does not necessarily equal $T'^{\mu\nu}$, and $S''^\mu$ does not necessarily equal $S'^{\mu\nu}$. But the relation between tensors $T$ and $S$ is preserved.

The plot thickens: the derivative of a tensor such as $\partial_\nu A^\mu$ (using a compact notation for $\partial_\mu A^\nu/\partial x^\nu$) does not respect the rules of tensor transformation! This spells trouble, because most physics principles are expressed as differential equations. Happily, the situation can be salvaged by enlarging the definition of the derivative, from $\partial_\nu$ into the “covariant derivative” $D_\nu$. For example, applied to a rank-1 tensor, $\partial_\nu A^\mu \to D_\nu A^\mu \equiv \partial_\nu A^\mu + \Gamma^\mu_\nu A^\nu$, where the Christoffel symbol (after Elwin Christoffel) or “affine connection” is a nontensor that may be written as

$$\Gamma^\mu_\nu = \frac{1}{2}g^{\mu\rho}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}), \quad (2)$$

with $g^{\mu\rho}$ a component of the multiplicative inverse of $g_{\nu\rho}$, so that by definition $g^{\mu\rho} g_{\rho\sigma} = \delta^\mu_\sigma$, which is the Kronecker delta (equal to 1 if $\mu = \nu$ and 0 if $\mu \neq \nu$). Tensor indices are raised and lowered with $g^{\mu\nu}$ and $g_{\nu\mu}$, e.g., $g^{\mu\nu}T^{\nu\rho} = T^\mu_\rho$ and $g^{\mu\nu}S_{\nu\rho} = S^{\mu}_{\rho}$. In a transformation, the terms that prevent $\partial_\nu$ and $\Gamma^\mu_\nu$ from separately being tensors cancel out in the covariant derivative $D_\nu$. Thus $D_\nu A^\mu$ transforms as a respectable rank-2 tensor, $(D_\nu A^\mu) = \Lambda^\rho_\nu A^\mu(D A^\rho)$. Covariant derivatives can be defined for tensors of higher rank, with upper or lower mixed indices. Significantly, the covariant derivative of the metric tensor always vanishes.

A generalization of Gauss’s curvature $K$ is found in the rank-4 Riemann curvature tensor, a kind of leftover residue from the commutator $D_\nu D_\mu - D_\mu D_\nu$, with components

$$R^\kappa_{\mu\nu\rho} = (\partial_\rho \Gamma^\kappa_\mu + \Gamma^\lambda_\rho \Gamma^\kappa_\mu_\lambda - (\partial_\lambda \Gamma^\kappa_\mu + \Gamma^\lambda_\mu \Gamma^\kappa_\rho_\lambda)). \quad (3)$$

For Einstein’s program, its contracted form, the Ricci tensor with components $R_{\mu\nu} = R^k_{\mu\nu\kappa}$, plays a central role. Now, back to the story…

The Einstein-Grossmann Collaboration

When Einstein arrived at the ETH and asked Grossmann about tensors and general covariance, according to one witness Grossmann answered that Riemannian geometry was needed, but he considered the Riemann tensor’s nonlinearity to be a disadvantage. However, Einstein knew his gravitation theory had to be nonlinear. Thus began the Einstein-Grossmann collaboration, where Grossmann introduced Einstein to tensor calculus and Einstein applied it to gravitation. Grossmann was happy to collaborate, but as a mathematician he inserted a caveat: “...He was ready to collaborate on this problem under the condition, however, that he would not have to assume any responsibility for any assertions or interpretations of a physical nature.”

The Einstein-Grossmann paper, “Outline of a Generalized Theory of Relativity and of a Theory of Gravitation,” was published in 1913. The Einstein-Grossmann collaboration, where Grossmann introduced Einstein to tensor calculus and Einstein applied it to gravitation. Grossmann was happy to collaborate, but
density $\rho$ according to Poisson’s equation,

$$\Delta \Phi = 4\pi G \rho, \quad (4)$$

where $G$ denotes Newton’s gravitational constant and $\Delta$ the Laplacian (using the notation fashionable during Einstein’s and Grossmann’s time).

The conservation laws would have to be respected, too. In Newtonian mechanics the local conservation of mass finds expression as an equation of continuity,

$$\nabla (\rho v) + \frac{\partial \rho}{\partial t} = 0, \quad (5)$$

where $v$ denotes velocity. In going from Newtonian to generally covariant gravitation, on the right-hand side of Eq. (4) the mass density $\rho$ would generalize to an energy-momentum tensor $T^{\mu\nu}$; precedents are found in hydrodynamics and electromagnetism. In the Lorentz covariance of special relativity, Eq. (5) generalizes to $\partial \rho T^{\mu\nu} = 0$, which expresses energy conservation for matter and electromagnetic fields. In general relativity this local conservation law generalizes further, through the covariant derivative, into

$$D_{\nu} T^{\mu\nu} = \partial_{\rho} T^{\rho\nu} + \Gamma_{\rho\nu}^{\sigma} T^{\sigma\nu} + \Gamma_{\nu\rho}^{\sigma} T^{\mu\sigma} = 0. \quad (6)$$

However, the $\Gamma T$ terms gave some interpretation problems to Einstein and his colleagues, as we shall see when our story continues in Part 3 of “The Journey Toward General Relativity.”

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References
[2] ETH = Eidgenössische Technische Hochschule, the Federal Institute of Technology. In 1895 Einstein failed his first go at the entrance exam to the ETH—there is hope for all students who struggle. See Abraham Pais, Subtle is the Lord: The Science and Life of Albert Einstein (Oxford University Press, 1982), 521.
[4] Steven Weinberg, Gravitation and Cosmology (Wiley, 1972), 8–11. [5] Pais 212. The same passage describes how in 1909, Max Born presented a paper on rigid body dynamics in special relativity that employed Riemannian geometry ideas. We do not know for certain whether Born’s presentation influenced Einstein’s thinking, but we do know that Einstein attended the conference.
[8] E.g., special relativity has the famous $t' = \gamma (t - vx/c^2)$ where $\gamma = (1 - v^2/c^2)^{-1/2}$, so that $\Lambda_{t'} = \partial t'/\partial t = \gamma$ and $\Lambda_{x'} = \partial t'/\partial x = -\gamma v/c^2$.
[10] Ibid. 219.