Physics Puzzler: The Physics of Chucking Pumpkins, Part 2 - rough solutions:

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A. Pumpkin Drop

By Newton's 2nd Law, $a = \frac{F}{M}$, where F is the total force on the pumpkin (Equ. 5) and M is the mass of the pumpkin and liquid nitrogen.

Based off the picture, the pumpkin appears to be a prolate spheroid. Figure 2 gives the volume and cross-sectional area of a prolate spheroid in terms of the major and minor radii, which we can calculate from the provided height and equatorial circumference.

$$c = \frac{1}{2} height = 15\ cm = .15\ m$$

$$a = \frac{circumference}{2\pi} = 10.35\ cm = .1035\ m$$

So, the volume and cross sectional area of the pumpkin are the following:

$$A = \pi a^2 = .0336\ m$$

$$V = \frac{4}{3}\pi a^2 c = .00673\ m^2$$

From the volume we can find the mass of the LN_2 within the pumpkin. The volume of the LN_2 is half that of the pumpkin, and the density of the LN_2 is provided in the question's hint. So,

$$mass = volume * density$$

$$m_{LN_2} = \frac{1}{2}.00673 * 808.4 = 2.72\ kg$$

The total mass of system is the mass of the pumpkin plus the mass of the LN_2, $M = 8.93kg$.

Finally, we can calculate the total acceleration. The mass-density and drag coefficient are given in the article. $g = 9.81\ m/s^2$.

$$a = \frac{Mg - \rho g h C_d A}{M} = 9.616\ m/s^2$$

Check your intuition: Since the drag force is opposing the force of gravity, the acceleration of the falling pumpkin should be slightly less than gravitational acceleration.

B. Devilish Differentials

Using Eq. 2 and Newton's 2^{nd} Law, and remembering that acceleration is the derivative of velocity, we find the following differential equation:

$$\frac{dv}{dt} = \frac{1}{m}(mg - \frac{1}{2}\rho v^2 C_d A)$$

Now we get to perform a trick that mathematicians hate. Let's divide both sides of the equation by everything on the right-hand side of the equal sign, then "multiply" by dt.

$$\frac{m * dv}{mg - \frac{1}{2}\rho v^2 C_d A dt} = dt$$

Taking the integral of both sides reveals an integrand that matches the identity given in the hint! Let us solve each side of the equation individually.

$$\int \frac{m * dv}{mg - \frac{1}{2}\rho v^2 C_d A dt} = \int dt$$

$$\int \frac{m * dv}{mg - \frac{1}{2}\rho v^2 C_d A dt} = \frac{m \tanh^{-1}\left[v\sqrt{\frac{2\,mg}{\rho C_d A}}\right]}{\sqrt{\frac{mg\rho C_d A}{2}}} + c_1$$

$$\int dt = t + c_2$$

We can solve for $v(t)$ by setting these solutions equal again.

$$\frac{m \tanh^{-1}\left[v\sqrt{\frac{2\,mg}{\rho C_d A}}\right]}{\sqrt{\frac{mg\rho C_d A}{2}}} + c_1 = t + c_2$$

The arbitrary constants c_1 and c_2 can be tossed out because of the initial condition that $v(t) = 0$ (the pumpkin was released from rest). Next we use a little algebra and take the hyperbolic tangent of both sides to solve for v.

$$\tanh^{-1}\left[v\sqrt{\frac{2\,mg}{\rho C_d A}}\right] = \frac{t}{m}\sqrt{\frac{mg\rho C_d A}{2}}$$

$$v\sqrt{\frac{2\,mg}{\rho C_d A}} = \tanh\left[\frac{t}{m}\sqrt{\frac{mg\rho C_d A}{2}}\right]$$

$$v(t) = \sqrt{\frac{\rho C_d A}{2\,mg}}\tanh\left[\frac{t}{m}\sqrt{\frac{mg\rho C_d A}{2}}\right]$$

And we've done it! This is definitely a stretch of some mathematical muscles. Now we can solve for the force on the pumpkin at specific times during its fall.