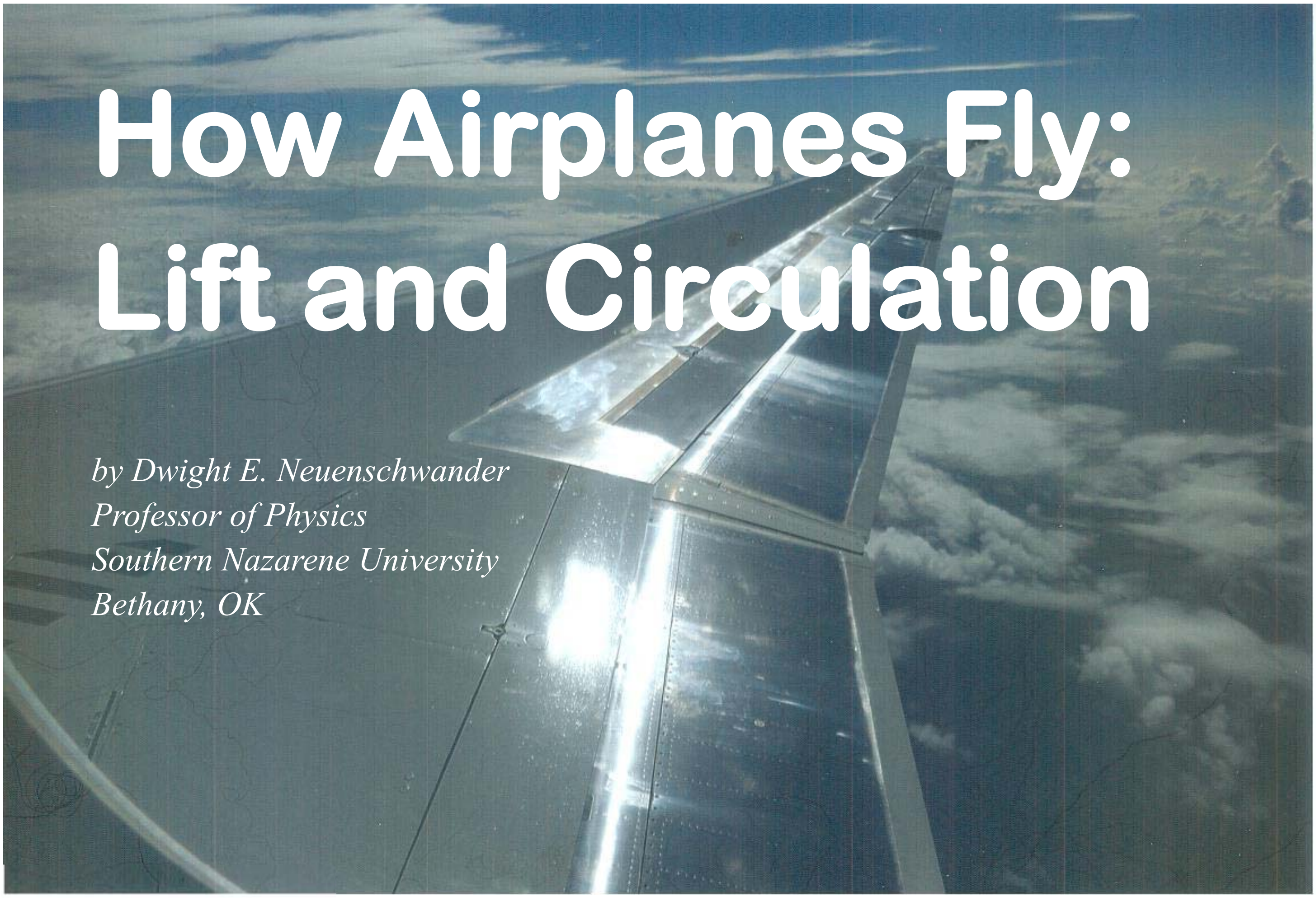

How Airplanes Fly: Lift and Circulation

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No matter how much physics you know, it's still amazing to see a 380,000 lb. airplane fly. Whenever I fly commercially, I request a window seat. The view of the landscape from on high is more interesting than any movie. The airplane's wings are fascinating to watch too—not only for what you see with your eyes, but also for what you see in your mind, thanks to physics.

When an airplane flies, the pressure P_l on the lower wing surface exceeds the pressure P_u on the upper surface. Equivalently, the air rushing by the lower surface moves slower than air passing over the upper surface. These statements are equivalent thanks to Bernoulli's equation—the work-energy theorem applied to fluids. For any two points 1 and 2 (Fig. 1) in non-turbulent, inviscid, incompressible, irrotational flow, Bernoulli says

$$(\tfrac{1}{2}\rho v^2 + \rho g y + P)_1 = (\tfrac{1}{2}\rho v^2 + \rho g y + P)_2 \qquad (1)$$

where ρ denotes the fluid's density and g the gravitational field. Since the top and bottom surfaces of a wing have essentially the same vertical coordinate y, it follows that $P_l > P_u$ if and only if $v_l < v_u$. But why should the air move faster over the top surface? Answering this question introduces the closed-path line integral of velocity, the "circulation" Γ,

$$\Gamma \equiv \oint_C \boldsymbol{v}\cdot d\boldsymbol{r} \qquad (2)$$

where the closed path C is traversed counter-clockwise. [1] Consider an airfoil—a wing's cross-section—in Fig. 1. The "chord length" L denotes the distance between the airfoil's leading and trailing edges. Assuming horizontal flow, the circulation evaluated over path ABCD gives $\Gamma = (v_l - v_u)L < 0$. Now *three* statements— $P_l > P_u$, $v_l < v_u$, and $\Gamma < 0$—are equivalent articulations of the condition necessary for lift.

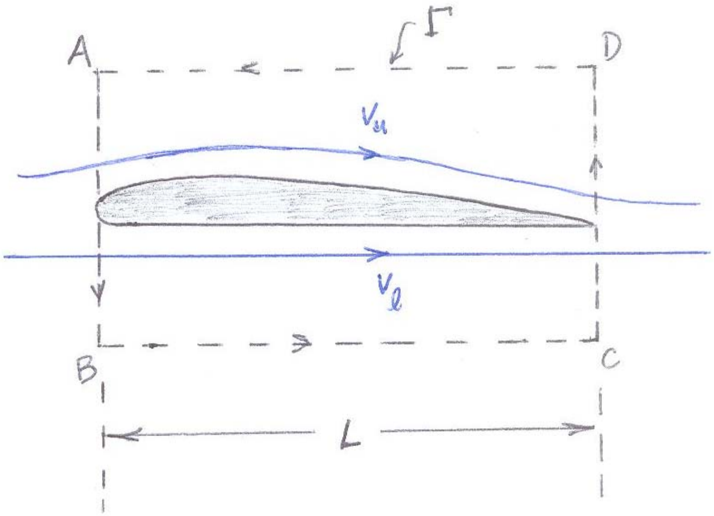

Fig. 1. The airfoil and geometry for Eq. (1).

Under conditions where Bernoulli's equation holds, $\Gamma = const.$ [2] This is "Kelvin's circulation theorem." From Eq. (2) for a fixed closed path we have

$$\frac{d\Gamma}{dt} = \oint_C \frac{d\boldsymbol{v}}{dt}\cdot d\boldsymbol{r}. \qquad (3)$$

Consider a parcel of air of volume $dV = Ady$. Its weight $dm\,\mathbf{g} = \rho\mathbf{g}Ady$ and the pressure force $[P(y) - P(y+dy)]A$ turn $d\mathbf{F} = dm\mathbf{a}$ into

$$-\frac{1}{\rho}\frac{dP}{dy}\hat{\boldsymbol{j}} + \mathbf{g} = \frac{d\boldsymbol{v}}{dt}. \qquad (4)$$

Eq. (4) shows the integral of Eq. (3) to be path-independent, proving the theorem. But this raises a puzzle. An airplane sitting in still air before takeoff has $\Gamma = 0$ for any closed path about an airfoil, but in flight $\Gamma < 0$. However, nonzero viscosity provides the loophole that lets planes fly while respecting Kelvin's theorem.

Inviscid fluids are idealizations—zero viscosity fluids would be infinitely runny. Water has a small viscosity, molasses a large viscosity. With viscosity, adjacent layers of fluid exert frictional forces on one another, tangent to the flow. On the wing surface the velocity is zero (driving your car does not sweep dust off it), and within a few millimeters above the surface the velocity reaches its free-stream value (Fig. 2). This region of velocity gradient is the "boundary layer." [3] Aircraft fuselages and wings taper to pointed trailing edges to prevent boundary layer separation from the surface, which would produce energy-consuming turbulence.

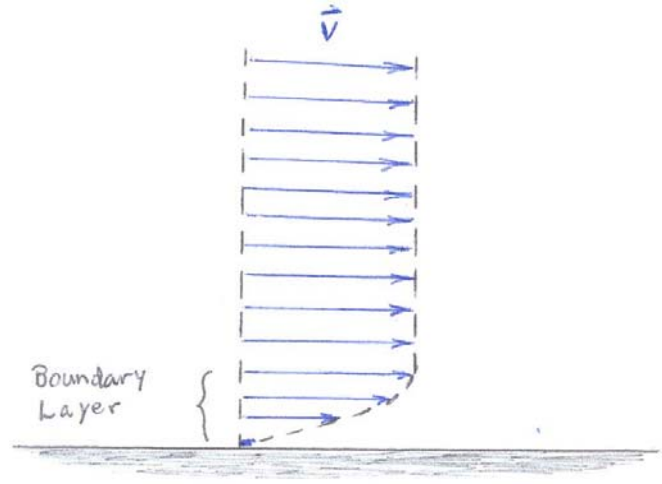

Fig. 2. Boundary layer due to viscosity.

Returning to the puzzle, a simple demonstration illustrates the crucial point. [4] Insert a ruler, held vertically, into a pan of still water. Suddenly push the ruler horizontally, such that its chord makes a non-zero "angle of attack" with the direction of motion. Watch the water at the trailing edge. Thanks to viscosity, a counter-clockwise vortex forms there (Fig. 3). [5]

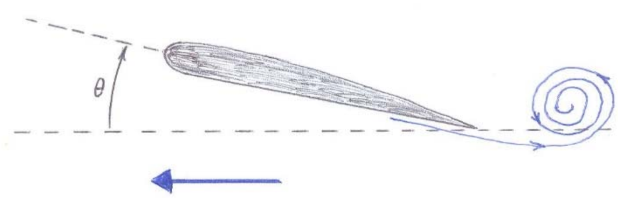

Fig. 3. The starting vortex.

Return to the airplane before takeoff, and consider the closed path ABHCDKA around the airfoil (Fig. 4). This path lies outside the boundary layer, where the effects of viscosity are negligible, so Kelvin's theorem applies. Before takeoff, $\Gamma_{ABHCDKA} = 0$. When the plane accelerates forward, the counter-clockwise vortex forms at the wing's trailing edge, within closed path HCDKH, so that $\Gamma_{HCDKH} > 0$. Consider path ABHKA that surrounds the airfoil but excludes the starting vortex (Figs. 4 and 5). Since $\Gamma_{ABHCDKA} = \Gamma_{ABHKA} + \Gamma_{HCDKH}$ (section

HK in Γ_{ABHKA} cancels KH in Γ_{HCDKH}), Kelvin's theorem requires $\Gamma_{ABHKA} < 0$. With the flows along ABHKA parallel to the chord, this requires $v_l < v_u$. By Bernoulli's equation this requires $P_l > P_u$, and the plane flies.

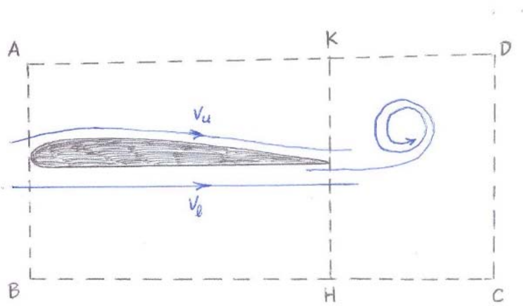

Fig. 4. The paths used in evaluating the circulation around an airfoil.

Fig. 5. The circulation path ABHKA.

From another perspective, a wing gets lift because it deflects the air's momentum downward (Fig. 6). Consider a parcel of air bouncing off a wing with nonzero angle of attack. [6]

Fig. 6. Wing with flaps down deflecting the air's momentum.

Before encountering the wing, the parcel approaches it with horizontal velocity v_∞. Compute $\Delta p_y/\Delta t$, which equals the net vertical force the wing exerts on the air parcel, which is opposite the upward force the air exerts on the wing. We have $\Delta p_y/\Delta t = (\Delta m/\Delta t)\, v_y$ where $\Delta m = \rho wh\Delta x$ for a parcel having height h, width w, and horizontal length $\Delta x = v_\infty \Delta t$. Thus $\Delta p_y/\Delta t = \rho hwv_\infty v_y$, and the upward force exerted on the wing by the air is $-\rho hwv_\infty v_y$ which is > 0 because $v_y < 0$. A vertical section of the closed path that includes v_y contributes $hv_y < 0$ to Γ. Thus the lifting force/length is $F_y/w = -\rho v_\infty \Gamma > 0$.

In 1902 Martin Kutta in Germany published "Lifting forces in flowing fluids," which related lift to circulation for 2-D flow past a circular arc with a trailing edge. In 1906 Nikolai Joukowski in Russia generalized the lift theorem, now called the "Kutta-Joukowski lift theorem," [7] relating circulation to the lift, perpendicular to $\mathbf{v}_\infty$, for any two-dimensional airfoil: $Lift/w = -\rho v_\infty \Gamma$. The value of Γ depends on the airfoil shape. A supplementary *ad hoc* Kutta-Joukowski hypothesis proposed a steady-flow value for Γ and thus a lift that agrees beautifully with phenomenology. [8] Specifically, for a thin symmetrical airfoil with a curved leading edge, sharp trailing edge, chord length L and angle of attack θ, the K-J hypothesis puts $\Gamma = -\pi v_\infty L\sin\theta$, so that $Lift/w = +\pi\rho(v_\infty)^2 L\sin\theta$. This result is to classical airfoil design what simple pendulums and Kepler's elliptical orbits are to other areas of mechanics: a model that serves as an excellent approximation to reality, and a starting-point for extensions to more complex situations. The K-J hypothesis works for θ less than 8 to 20 degrees depending on airfoil shape; at higher angles boundary layer separation on the upper surface causes a stall. [9]

On December 17, 1903, the Wright brothers achieved the first successful powered flight. They did not use the lift theorem, but took an empirical hands-on approach. Their airfoil designs were guided by data they collected from their wind tunnel. In 1909 Wilbur Wright wrote, "…I think it will save me much time if I follow my usual plan and let the truth make itself apparent in actual practice." [10] The Kutta-Jukowski hypothesis for Γ, along with their namesake lift theorem, affirms theoretically what the Wrights found experimentally.

The next time you fly commercially, ask for a window seat, and *really look* at the wing. Imagine the circulation around the airfoil generating the pressure differential that keeps the airliner aloft. In humid weather this circulation is sometimes made visible to the eyes (Fig. 7) as well as to the mind—because the wing has finite length, this circulation can show up as vortex of water vapor spinning off the wing-tips. [11] *Bon voyage!*

Fig. 7. Humidity making the flow visible. The condensation above the wing shows the pressure to be lower there.

Acknowledgments

[1] Familiar closed-path integrals include those of force (work) and electric and magnetic fields (Faraday and Amperé laws). One may wonder how a velocity field can be "irrotational" and have non-zero circulation. By Stokes' theorem, Γ equals the flux of the curl of $\boldsymbol{v}$. "Irrotational" means the curl is everywhere locally zero, even though there may be large-scale circulation.

[2] D.J. Acheson, *Elementary Fluid Dynamics* (Clarendon Press, Oxford, UK, 1990), 157-157 offers a proof in three dimensions.
[3] *ibid.,* 26-28.
[4] *ibid.,* 1-2, for a more sophisticated version of this table-top demonstration.
[5] If your ruler has a sharp leading edge, a vortex forms there too. A leading vortex does not occur for airfoils with rounded leading edges.
[6] Flaps and curvature of the wing surfaces produce the momentum transfer even with zero angle of attack.
[7] Acheson, ref. 2, Ch. 4.
[8] *ibid.,* 20-21. The K-J hypothesis is *ad hoc* but not arbitrary, motivated by considering 2-D flow past a cylinder, whose velocity field can be found exactly from the Navier-Stokes equation ($\mathbf{F} = m\mathbf{a}$ applied to a viscous fluid). One then carries out a conformal transformation to distort the cylinder into an airfoil with a sharp trailing edge, and the velocity field is transformed too. The K-J hypothesis ensures a negative Γ without the velocity becoming infinite at the trailing edge.
[9] *ibid.,* 29-30.
[10] *ibid.,* 121.
[11] *ibid.,* 22-23. Vortices trailing off the wingtips can be seen when the air is humid and the pressure change causes condensation, giving a rope-like vortex spinning off the wingtip. Mathematically these occur because, by Stokes' theorem, Γ can be written as the flux of the curl of $\boldsymbol{v}$ through *any* surface that has closed path C for its boundary. That surface can go around the wingtip like a butterfly net, and vorticity must pass through that surface.

All photos courtesy of the author.