Thin Lenses and Chromatic Aberrations

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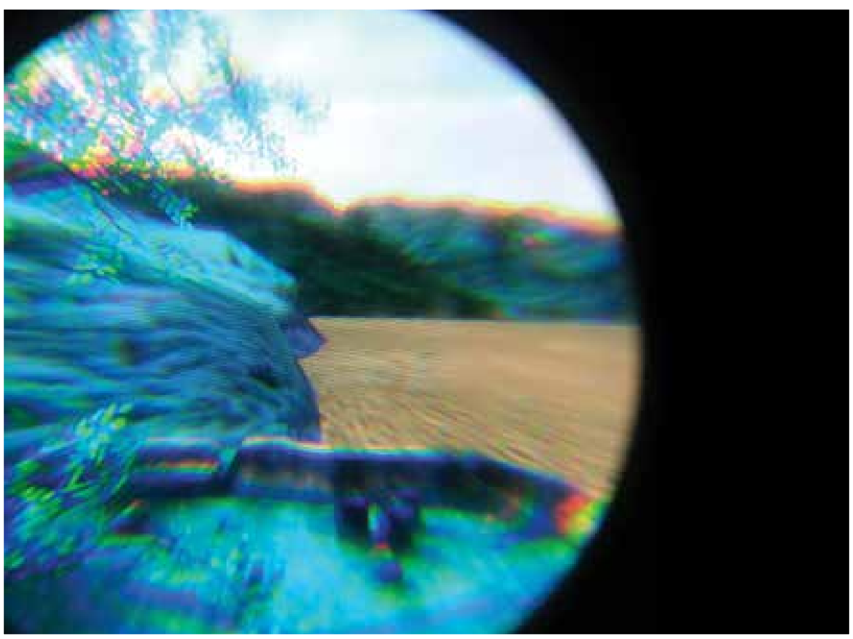
As SPS gears up to participate in the 2015 UNESCO (United Nations Educational, Scientific, and Cultural Organization) Year of Light and Light-Related Technologies, we are reminded of the old proverb that says, "The eyes are the window to the soul." If through the window to your soul you look closely at a small purple dot on a white background, it appears blue in the center surrounded by red. Move farther out, and the dot appears red in the center surrounded by blue.[1] I do not know what this means for the soul, but for optics it means that the lens in the human eye exhibits *chromatic aberration*. This effect occurs when the index of refraction of the lens material varies with the wavelength of light. Such *dispersion* separates light into a rainbow when passing through a prism and causes interesting challenges for the designers of refractive optical systems.

YOUR MISSION IS TO DESIGN A compound lens.



A HEAVY CHROMATIC ABERRATION occurs in a virtual reality software demonstration. Image credit: Filip Hajek/Bill Cummings.

Imagine you, a physicist, have been hired as an optical engineer for a camera manufacturing company. You have been given the task of designing a system of lenses that will make red light and blue light come to a focus at the same spot. Of course, there are other colors in the spectrum besides red and blue. But by building a compound lens that focuses onto the same spot light from opposite ends of the visible spectrum, you will have made an important step toward removing most of the chromatic aberration.

When Isaac Newton began making his own telescopes, he started with a refractor and set to work grinding his own lenses. However, Newton was dismayed to see that red and blue light did not focus at the same place. He gave up on refractors and proceeded to invent the telescope that today bears his name, the Newtonian reflector. It remained for Chester Moore Hall, a British lawyer and inventor, to invent in 1733 the first achromatic compound lens (a two-lens system) that he used in his own telescope. In 1758 the English optician John Dolland reinvented and patented the achromatic compound lens.[2] Today you, dear reader, are invited to design an achromatic lens system using nothing but the equations developed in the theory of thin lenses, which most physics students meet in introductory physics class—well, using not quite "nothing but" thin-lens formulas. As we shall see, we also need the trick of adding a third wavelength, which opens to us some tabulated parameters for glass called Abbe numbers; but these, too, are defined by expressions that follow from thin-lens equations. The Abbe numbers provide constraints that reduce otherwise wild ambiguity in the parameter space. It's all introductory optics, but applied in a clever way.

The simplest model of image formation by a lens is expressed in the well-known thin-lens equation coupled with the lens maker's equation. These working equations for imaging with lenses are derived by applying Fermat's principle to refraction at a spherical surface, then using the image formed by one spherical surface as the object for a second spherical surface.[3] The "thin-lens" approximation comes by neglecting the thickness of the lens, assumed to be small compared to other distances in the problem. In the following discussion I assume the reader to be familiar with thin-lens optics. (If you are not, see any general physics or optics textbook, or Ref. 3.) The thin-lens equation relates the distance i of the image from the lens to the object distance o and the focal length f, according to

$$1/o + 1/i = 1/f, \tag{1}$$

where f is the focal length of the lens, which depends on the lens material's index of refraction n and the radii of curvature of the lens surfaces. These define f in the lens maker's equation, which emerges as the right-hand side in the derivation of Eq. (1) according to

$$1/f = (n - 1)(1/a - 1/b). \tag{2}$$

In Eq. (2) a denotes the radius of curvature of the first surface encountered by the light in passing through the lens and b the radius of curvature of the second surface. Although distance and radii are normally non-negative numbers, in lens physics the image distance and radii of curvature may be positive or negative, depending on whether the image and the centers of curvature lie on the real side or on the virtual side of the lens (the real side is the side where the light energy really is after encountering the lens, and the virtual side is the opposite side). By Eqs. (1) and (2) the image location is given by

$$1/i = (n - 1)W - 1/o, \tag{3}$$

where the lens surface curvatures have been lumped into the term

$$W \equiv 1/a - 1/b\ . \tag{4}$$

Chromatic aberration occurs when the refractive index is a function of the wavelength λ of light, $n = n(\lambda)$. If red and blue are present together in the light ray, then the focal length for red will differ from the focal length for blue. This is the problem that you, the camera lens designer, are to address. The aberration produced by one lens for incoming light of two wavelengths can be compensated for by having the light pass through a suitably designed second lens before the final image forms. As lens physicists let us work out the simplest correction to chromatic aberration.

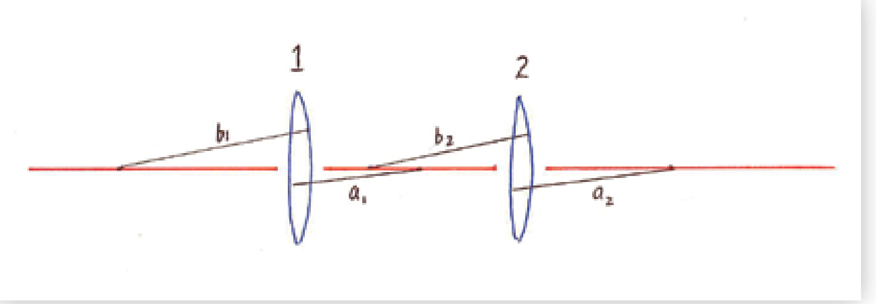


FIG. 1 The two-lens system and the radii of curvature of the first and second lenses.

Consider a beam of light that contains two wavelengths, λ and λ'. (When it's time to insert numbers, we will take for red the wavelength $\lambda = 6562.816$ Å and for blue the wavelength $\lambda' = 4861.327$ Å. But we will leave the equations in terms of symbols as long as possible so that the results derived will apply to any two wavelengths and any pair of lenses.) Now introduce a second lens coaxial with the first. For illustrative clarity (see Fig. 1), let the two thin lenses be separated by some distance D; in our compound lens application we will bring the lenses together by setting $D = 0$. To distinguish variables belonging to the two lenses, let us use subscripts 1 and 2 for them, and let parameters that depend on the two wavelengths be primed and unprimed. For instance, when the light passes through lens 1 the index of refraction for the red wavelength will be $n_1(\lambda) \equiv n_1$. When the blue wavelength passes through lens 1, the refractive index will be denoted $n_1(\lambda') \equiv n_1'$. Similar notations hold for lens 2.

Your mission is to design a compound lens, two lenses back to front ($D = 0$), such that after passing through both lenses the rays corresponding to both wavelengths form final images in the same place. A solution is worked out in an Appendix on p. 31, but please give it your best shot before looking there. The path toward solving the problem may be separated into four steps. In doing step (1) you will cross the main conceptual threshold for solving the problem. Its result provides the essential relationship between the two lens' refractive indices and their radii of curvature that will get the job done. However, this is one equation with many unknowns. To converge on a specific set of numerical values for these parameters without having to make too many arbitrary assumptions, we will consider other constraints. The guided moves for carrying us farther are outlined in steps (2)–(4). Here we go . . .

01 Derive an equation that relates the radii of curvature and indices of refraction of the two lenses so that light of wavelengths λ and λ' form their final images at the same location (with $D = 0$). Recall that in any multiple lens system the image produced by the first lens becomes the object for the second lens. If the image produced by the first lens lands *behind* the second lens, that means the first image never actually forms; the rays are redirected by the second lens. But one solves the problem by finding where the first lens *would have* formed the image, then making that site the object location for the second lens.

02 Show that for the two-lens system, the effective focal length f_{eff} which relates the *original* object to the *final* image (again with $D = 0$) is given by

$$1/f_{eff} = 1/f_1 + 1/f_2. \tag{5}$$

03 Write the result of exercise (1) as an expression for W_2/W_1. Set that aside temporarily and consider light of a third wavelength λ'' passing through the compound lens system. Derive another expression for W_2/W_1 in terms of f_1'', f_2'', n_1'', and

n_2''. Set the two expressions for W_2/W_1 equal to show that

$$f_1'' V_1 + f_2'' V_2 = 0, \tag{6}$$

where

$$V_1 = (n_1'' - 1)/(n_1' - n_1), \tag{7}$$

and similarly for V_2. These *V parameters* or *Abbe numbers* for various glasses are well cataloged by glass manufacturers. If a glass existed for which the refractive index was the same for all wavelengths, there would be no dispersion and the Abbe number for that hypothetical glass would be infinite. Thus low dispersion corresponds to large Abbe numbers, and highly dispersive glasses have low Abbe numbers. Glass physicists use three of a standard set of spectral lines, the so-called *Fraunhofer lines,* to measure and catalog the Abbe numbers.[4] Once the types of glass are selected from which the two lenses will be made, their Abbe numbers are known from the tabulated data. Then Eqs. (5) and (6) (both applied to light of wavelength λ'') give a pair of simultaneous equations that can be solved for the values of f_1'' and f_2''. After these individual focal lengths are known, we can choose one radius of curvature for one of the lenses and apply Eq. (2) to each lens, which tells us the remaining radii of curvature needed to grind both lenses.

04 Now we are ready to design (with numbers) a specific achromatic compound lens. The two wavelengths we require to arrive at a common image we take to be the red Fraunhofer C line $\lambda = 6562.816$ Å and the blue F line, $\lambda' = 4861.327$ Å.

Next we select an effective focal length for our compound lens. Suppose the lens you are designing will be a telephoto lens. Telephoto lenses offer a narrow field of view but bring a faraway subject in close. Their focal lengths range from 100 to about 800 mm. Let us choose our telephoto compound lens system to have an effective focal length of $f_{eff} =$ 300 mm.

Next we must choose the lens materials. Let us choose crown glass for lens 1, for which the glass catalogs say $V_1 = 59.6$. For lens 2 we choose extra dense flint (EDF), for which $V_2 = 30.9$.[5]

From these choices show from Eqs. (5) and (6) that $f_1'' = 14.45$ cm and $f_2'' = -27.86$ cm. Then from Eq. (2) the radii of curvature of each lens can be determined. Let lens 1 be an equiconvex lens (i.e., $a_1 = |b_1|$). Show that $a_1 = 14.97$ cm, and for lens 2 (which fits snug against lens 1) show that $b_2 = -66.67$ cm.

Now you have the specifications you need to grind the two lenses out of crown and EDF glass. They can then be glued together to make the achromatic compound lens. (We neglect any optical properties of the glue.) For solution details see the Appendix on p. 31. //

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In proclaiming an International Year focusing on the topic of light science and its applications, the United Nations has recognized the importance of raising global awareness about how light-based technologies promote sustainable development and provide solutions to global challenges in energy, education, agriculture and health. Light plays a vital role in our daily lives and is an imperative cross-cutting discipline of science in the 21st century. It has revolutionized medicine, opened up international communication via the Internet, and continues to be central to linking cultural, economic and political aspects of the global society.

ACKNOWLEDGMENTS

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REFERENCES

[1] Eugene Hecht, *Optics,* 4th ed. (Pearson/Addison-Wesley, San Francisco, CA, 2002), p. 269, suggests this exercise, which has been verified by the author. Hecht's book is referenced throughout this article. However, this topic is an old one, well covered in many optics books, for example, Francis A. Jenkins and Harvey E. White, *Fundamentals of Optics* (McGraw-Hill, New York, NY, 1950), pp. 145–153; Bruno Rossi, *Optics* (Addison-Wesley, Reading, MA, 1965), pp. 87–90.

[2] Hecht, Ref. 1, p. 271.

[3] "Elegant Connections in Physics, Foundations of Geometrical Optics: Phenomenology and Principles," *The SPS Observer* (Summer 2010), http://www.spsobserver.org.

[4] The *V* numbers take on the proper noun designation *Abbe numbers* when $\lambda'' =$ 5892.9 Å (the Fraunhofer D line halfway between the yellow sodium doublet); when $\lambda' = 4861.327$ Å, the Fraunhofer F line from helium; and when $\lambda = 6562.816$ Å, the Fraunhofer C line emitted by hydrogen. The Abbe number is named after its originator, the German physicist Ernst Abbe (1840–1905). See Hecht, Ref. 1, pp. 269–270.

[5] *Ibid.,* p. 270.

Solution to the Achromatic Compound Lens Problem

Here is a solution to parts (1)–(4) of the achromatic compound lens design problem from p. 24–26.

01 According to the thin-lens equation, monochromatic light impinging on lens 1 forms an image at i_1 given by

$$1/i_1 = 1/f_1 - 1/o_1. \qquad (S1)$$

The image of lens 1 becomes the object for lens 2. Therefore $o_2 = D - i_1$ so that $o_2 = -i_1$ when we set $D = 0$. (Recall that the second object is located where the first image *would have* formed.) The final image lands at i_2 given by

$$1/i_2 = 1/f_2 + 1/i_1$$

$$= 1/f_2 + 1/f_1 - 1/o_1$$

$$= (n_2 - 1)W_2 + (n_1 - 1)W_1 - 1/o_1. \qquad (S2)$$

Now let two wavelengths, λ and λ′, be present in the light beam. Write Eq. (S2) twice, once for λ and once for λ′. To remove their chromatic aberration we require $i_2 = i_2'$, which gives

$$(n_2' - n_2)W_2 = (n_1 - n_1')W_1. \qquad (S3)$$

When writing out explicitly the radii of curvature of the lens surfaces, Eq. (S3) becomes [1]

$$(n_2' - n_2)(1/a_2 - 1/b_2)$$

$$= (n_1 - n_1')(1/a_1 - 1/b_1). \qquad (S4)$$

Since the two lenses are glued together back to front, $b_1 = a_2$. Suppose lens 1 to be given (i.e., its refractive indices for both wavelengths and its radii of curvature are known). Then the right-hand side of Eq. (S4) is a known constant. On the left-hand side of Eq. (S4) we can play around with various values of b_2 and the type of glass for lens 2, giving many scenarios that would satisfy Eq. (S4). For a simple illustration of using Eq. (S4) as it stands, suppose lens 1 is plano-convex, so that $b_1 = \infty$. Because the lenses are back to front, we must also write $a_2 = \infty$. Equation (S4) then reduces to

$$(n_2' - n_2)/b_2 = (n_1' - n_1)/a_1. \qquad (S5)$$

Steps (3) and (4) will show how the multiplicity of possible lens parameters can be systematically reduced by introducing a third wavelength.

02 Whether or not two coaxial thin lenses are designed to be achromatic, together they are equivalent to a single lens of focal length f_{eff}. To find this effective focal length, return to the derivation of Eq. (S2) and note the intermediate step

$$1/o_1 + 1/i_2 = 1/f_2 + 1/f_1. \qquad (S6)$$

This can be written as a thin-lens equation for light from object 1, winding up as image 2 if we identify

$$1/f_2 + 1/f_1 = 1/f_{\text{eff}}. \qquad (S7)$$

03 Now let us turn to the role of the auxiliary wavelength λ″. With this color the two lenses have focal lengths given by Eq. (2), so that $1/f_1'' = (n_1'' - 1)W_1$ and similarly for lens 2. Taking the ratio W_2/W_1 in this instance and setting this ratio equal to that obtained from Eq. (S4) gives, after some algebra, Eq. (6), with the V numbers, or Abbe coefficients, defined by Eq. (7).

04 Equations (5) and (6) can be solved simultaneously for f_1'' and f_2''. In terms of symbols we obtain

$$f_1'' = f_{\text{eff}}''(V_1 - V_2)/V_1 \qquad (S8)$$

and

$$f_2'' = -f_{\text{eff}}''(V_1 - V_2)/V_2. \qquad (S9)$$

The value of $V_1 - V_2$ is typically chosen to be large to avoid a tight radii of curvature for the lenses. For crown glass the tabulated data says $V_1 = 59.6$ and $V_2 = 30.9$ for EDF (see footnote 5, p. 26). Upon choosing $f_{\text{eff}} = 30$ cm, Eqs. (S8) and (S9) give $f_1'' = 14.45$ cm and $f_2'' = -27.86$ cm. Choosing lens 1 to be an equiconvex lens means that $b_1 = -a_1$, and Eq. (2) gives $a_1 = 14.97$ cm. Since the two lenses are glued together, the first surface of lens 2 has the same radius of curvature as the second surface of lens 1 so that $b_1 = a_2$. Equation (2) then tells us that $b_2 = -66.67$ cm. Now you have all the specifications you need to grind your lenses!

Let us reflect on the strategy here: We require the red and blue wavelengths to form their images at the same spot. But there are too many free parameters. A pair of lenses placed back to front presents us with two indices of refraction and three radii of curvature. After choosing the glass materials, the radii of curvature remain unspecified. To help us select specific shapes for lenses 1 and 2, we bring in a third color whose wavelength "splits the difference" between red and blue. After we're finished, the red and blue wavelengths selected for achromatic treatment will be brought to a common focus, but other wavelengths will not. The residual chromatic aberration is called the *secondary spectrum.* Upon looking at a distant white object through a set of binoculars, the secondary spectrum can be faintly seen as a border in green and magenta around the image edges. [2] Compound systems of three or more lenses have been built for specialized tasks that require the secondary spectrum to be further minimized. (Real lenses in high-dollar cameras are not perfectly spherical either, but thanks to computer ray tracing that finesses the details for different wavelengths, their surfaces can be described by eighth-order polynomials!)

To celebrate your successful design of an achromatic lens system, I suggest brewing yourself a cup of fine coffee and going outside to watch the sunset. Take your camera, too. It probably has an achromatic compound lens. As you sip your coffee and watch the sky, Alexander Pope offers for your meditations a poetic background for the challenges in bringing diverse colors to a common focus:[3]

Dipt in the richest tincture of the skies,
Where light disports in ever-mingling dyes,
While ev'ry beam new transient colors flings,
Colors that change whene'er they wave their wings. //

REFERENCES

[1] For another derivation see Hecht, Ref. 1, p. 269. Our Eq. (S3) is his Eq. (6.47).
[2] Hecht, Ref. 1, p. 273.
[3] Lines from Canto II of "The Rape of the Lock" (1712, expanded 1714), by Alexander Pope (1688–1744). Pope was a contemporary of Newton. This poem is a social satire, told with the pomposity of a Greek epic, about a trivial incident of a lock of hair being snipped off without permission.