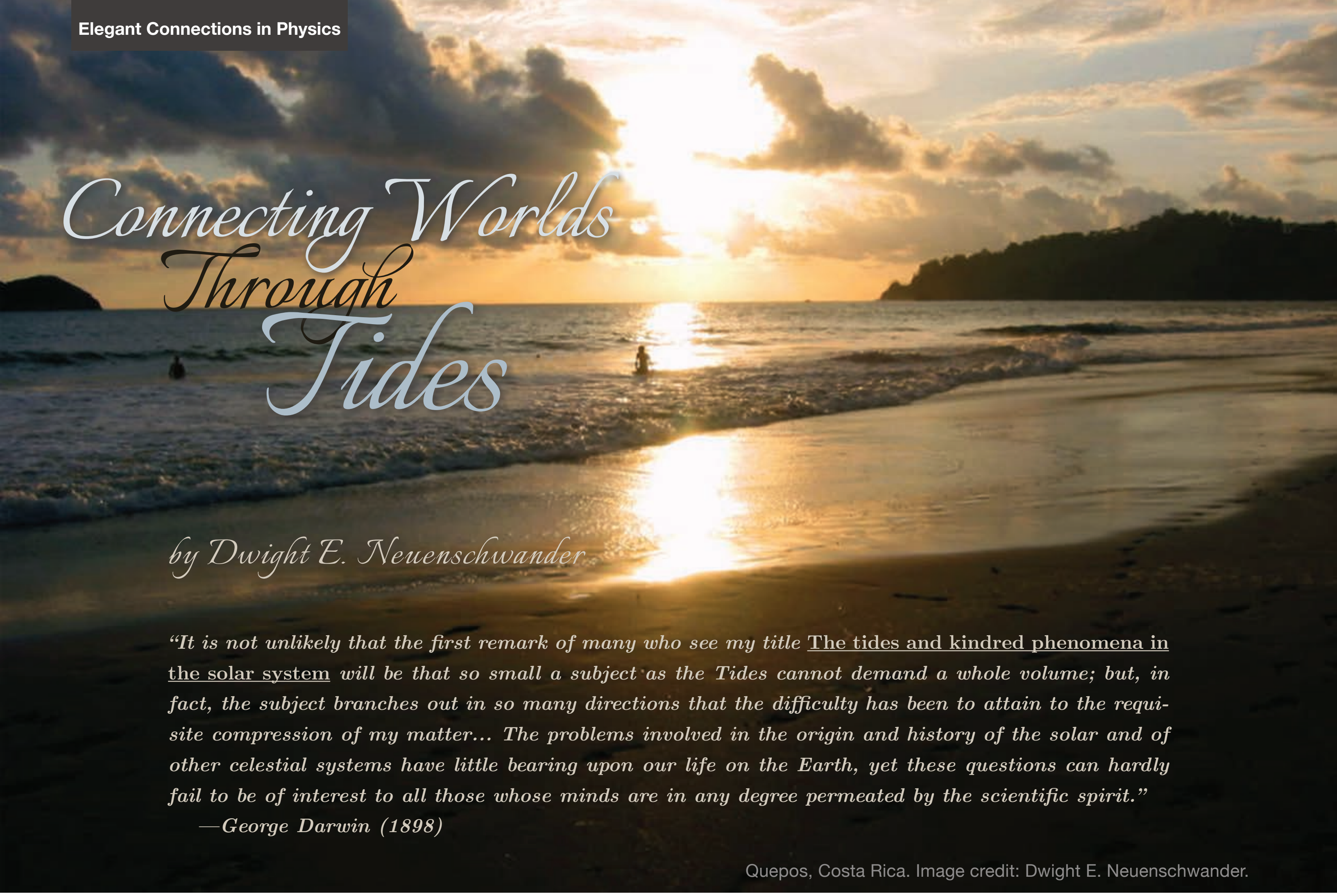

Connecting Worlds Through Tides

by Dwight E. Neuenschwander

"It is not unlikely that the first remark of many who see my title The tides and kindred phenomena in the solar system *will be that so small a subject as the Tides cannot demand a whole volume; but, in fact, the subject branches out in so many directions that the difficulty has been to attain to the requisite compression of my matter... The problems involved in the origin and history of the solar and of other celestial systems have little bearing upon our life on the Earth, yet these questions can hardly fail to be of interest to all those whose minds are in any degree permeated by the scientific spirit."*
—*George Darwin (1898)*

Quepos, Costa Rica. Image credit: Dwight E. Neuenschwander.

Our obligation to the Sun as the source of energy for life on this planet needs no elaboration. Should our appreciation also extend to the Moon? A journalist once said, "We live with the Moon, but not by it."[1] That may be true of our attitudes toward our Moon and our awareness of it, but if the Moon did not exist, how different would life on this planet be?[2]

Astronomical bodies are connected by gravitation, which holds satellites in their orbits about stars and planets. To understand the orbits themselves we initially model these bodies as point masses. But planets and moons have finite sizes. Gravitational forces vary with distance, and gradients in the field produced by one body generate internal stresses in other bodies. For instance, the Moon's gravity is slightly stronger on the Moon-facing side of Earth than on the opposite side, producing strains throughout Earth, the "tidal forces." The distortions produced by these stresses show up most dramatically in the oceans, forming two bulges of water on opposite sides of the planet. The bulges lie approximately on the line connecting the centers of the Earth and the Moon.

The bulge on the side of the Earth facing the Moon makes intuitive sense, but why is there a bulge on the *opposite* side? The authors of one introductory astronomy textbook write that the second bulge exists "because the Moon pulls more strongly on the Earth's center than on the far side. Thus the Moon pulls the Earth away from the oceans, which flow into a bulge away from the Moon...."[3] Other authors write that "the high tide on the far side of the Earth from the Moon occurs because the water there is 'left behind' as the Moon pulls the solid center of the Earth toward it."[4] Such explanations were written for nonmathematical audiences, so the authors had to resort to plausibility arguments. "Elegant Connections" readers may prefer discussions linked directly to elementary but quantitative physics. Let's have a go at it.

To get at the essentials of the tidal mechanism and the appearance of two tidal bulges, we will begin with an oversimplified model of the Earth. Then we will add realistic complications, including friction created by the Earth's crust rotating beneath the oceans. The friction acts like a brake applied to Earth's spin and slightly displaces the bulges from the Earth–Moon axis. This offset allows Earth and the Moon to exert torques on one another. The consequences include Earth's day gradually getting longer as the Moon slowly recedes from Earth.

Geological and paleontological evidence has long supported the hypothesis that the days were shorter in the distant past. To clinch the the torque-coupling hypothesis, confirmation was needed that the Moon's orbit is receding. This became possible in 1969, when the Apollo 12 astronauts left reflectors on the lunar surface, enabling the distance between Earth and the Moon to be measured to millimeter precision with pulsed lasers.

A Simple Model of Tidal Forces and Bulges

Let us start with an isolated, smooth, spherically symmetric, nonrotating Earth with a rocky core covered by a global ocean. Such an Earth is necessarily spherical because gravity is an inverse-square central force.[5]

Now bring the Moon into the game. Assume it to be a spherically symmetric body. So long as Earth and the Moon can each be modeled as a single particle, tides do not enter into the picture. Each body orbits the system's center of mass, following Kepler's laws; equivalently, the system behaves like a "reduced mass" $mM/(m+M)$ orbiting in a central field.[6]

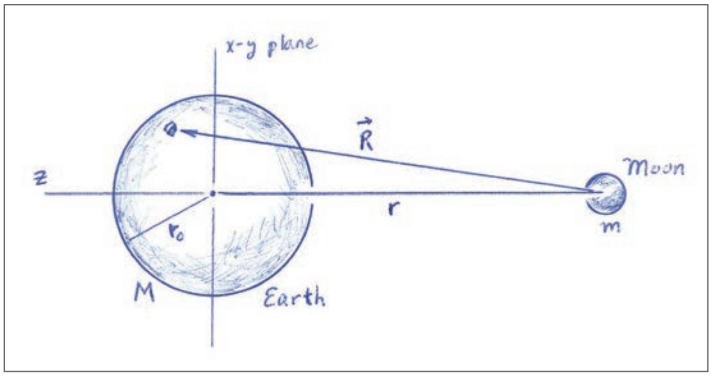

Fig. 1. Coordinate system for calculating tidal forces. The origin lies at Earth's center, with the Moon located distance *r* from it.

For now, let's ignore the effects of the other planets and the Sun. Once we understand how one body produces tides on another, we can include the tidal effects of additional bodies by superposition. We will also pretend for now that the Moon orbits directly over Earth's equator.

The Moon exerts a gravitational force on Earth. To a first approximation the Moon can be considered a point source of a gravitational field **g**. We need to map this **g** across Earth. As observers riding on Earth, let us fix our coordinates to it.

Let the unperturbed spherical Earth have mass M and radius r_o, label the Moon's mass m, and denote the distance between the centers of the bodies r. Place the origin of an xyz coordinate system at the center of Earth. The displacement from the Moon's center to an arbitrary point within Earth's volume is described by vector **R** (Fig. 1) [7]:

$$\mathbf{R} = x\,\mathbf{i} + y\,\mathbf{j} + (z+r)\mathbf{k}. \tag{1}$$

The Moon's gravitational field at that point is $\mathbf{g}(x,y,z) = -Gm\mathbf{R}/R^3$. Since $r \approx 60r_o$, to first order in ratios such as x/r we can use the binomial theorem to write

$$\frac{1}{R^3} \approx \frac{1}{r^3}\left(1 - \frac{3z}{r}\right) \tag{2}$$

so that

$$\mathbf{g}(x,y,z) \approx -\frac{Gm}{r^3}\,[x\mathbf{i} + y\mathbf{j} + (r - 2z)\mathbf{k}]. \tag{3}$$

We must also note that Earth accelerates toward the Earth–Moon system's center of mass. Whenever we do physics in a reference frame that accelerates with acceleration $\mathbf{a}_o$ relative to an inertial frame, we must subtract $\mathbf{a}_o$ from the acceleration **a** that is due to the "real" forces. Without the tides or Earth's rotation, *every* point in the planet would fall freely toward the Moon with an acceleration equal to the Moon's gravitational field at Earth's *center*, $\mathbf{a}_o = (Gm/r^2)(-\mathbf{k})$. Therefore, the *effective* lunar gravitational field acting on Earth, $\mathbf{g}_{eff} = \mathbf{g} - \mathbf{a}_o$, becomes

$$\mathbf{g}_{eff}(x,y,z) = \frac{Gm}{r^3}\,[-x\mathbf{i} - y\mathbf{j} + 2z\mathbf{k}]. \tag{4}$$

A map of $\mathbf{g}_{eff}$ throughout Earth's volume resembles a quadrupole, pushing Earth outward along the Earth–Moon axis (our z axis)

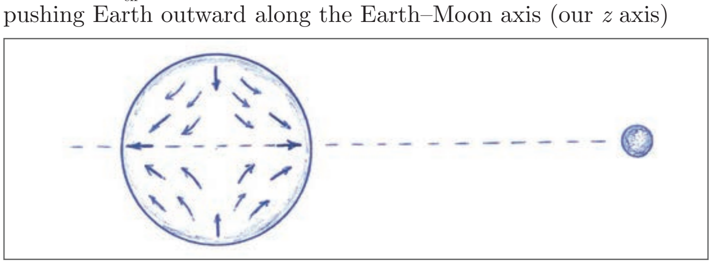

Fig. 2. Tidal stresses on the Earth due to the Moon.

and squeezing inward in cross sections perpendicular to the axis (Fig. 2). At a point on Earth's surface along the z axis, $|g_{eff}| = 1.1\times10^{-6}$ m/s^2, about 10^{-7} of Earth's own gravitational field.[8] This difference and the planet's plasticity allows its material to bend or flow in response to stresses. A bulge forms facing the moon, and another bulge exists on the opposite side.

If tidal forces exceed a body's self-gravitation and internal cohesive forces, the tides rip the body apart. The closest orbital radius about which one body can orbit another without disintegrating is called the "Roche limit." I will return to this below.

As Earth spins under the Moon, a wave of flexing a few centimeters in amplitude runs through the lithosphere. We who are on land bob up and down as we are carried along. The oceans have a more noticeable response, exhibiting pronounced tidal bulges. From where we sit on the seashore, we see the tide "come in" until high tide, then "go out" toward low tide. About $12\frac{1}{2}$ hours later our piece of coastline passes through the second high-tidal bulge (two per day, the "diurnal" tides).[9] Why $12\frac{1}{2}$ and not 12 hours? Earth spins west to east, so the Moon appears to rise in the east and set in the west. However, it rises about 52 minutes later each day, because it orbits eastward about 12° daily. Following the Moon,

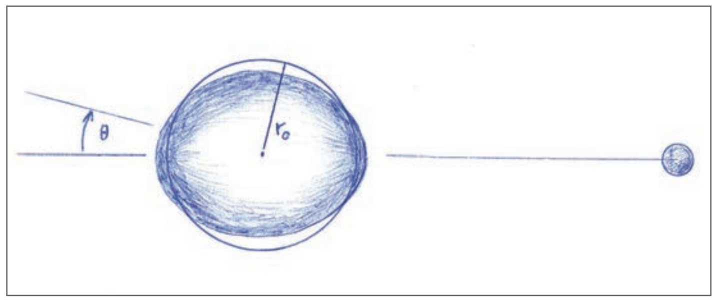

Fig. 3. Tidal bulges.

the tidal bulges creep eastward across Earth's surface.

We can calculate the shape of an idealized ocean surface with tidal bulges (neglecting Earth's rotation, solar tides, and coastline effects). Think of $\mathbf{g}_{eff}$ as the negative gradient of a potential φ. The potential that yields $\mathbf{g}_{eff} = -\nabla\varphi$ is

$$\begin{aligned}\varphi &= -\frac{Gm}{r^3}\left(-\frac{x^2}{2} - \frac{y^2}{2} + z^2\right) + const.\\ &= -\frac{Gmr_o^2}{2r^3}\,(3\cos^2\theta - 1) + const.\end{aligned} \tag{5}$$

where, on the surface, $r_o^{\,2} = x^2 + y^2 + z^2$ and $z = r_o\cos\theta$ (Fig. 3). A sailboat of mass m' floating on the ocean surface "feels" not only Earth's own gravitation, but also the potential energy of φ. At an elevation of $h+r_o$ above Earth's center, the sailboat's potential energy, for $h << r_o$, is

$$U = m'g_oh + m'\varphi, \tag{6}$$

with $g_o = GM/r_o^{\,2}$ denoting the magnitude of Earth's own gravitational field at r_o.[8] An ideal ocean in equilibrium has an equipotential surface with an axially symmetric but oblong shape,

$$h = \frac{GMr_o^2}{2g_or^3}\,(3\cos^2\theta - 1) + const. \tag{7}$$

High tides occur at $\theta = 0°$ and 180°, along the line connecting the terrestrial and lunar centers. Low tides occur at $\theta = \pm 90°$. The maximum and minimum heights differ by

$$\Delta h = \frac{3Gmr_o^2}{2r^3g_o} = \frac{3mr_o^4}{2Mr^3}. \tag{8}$$

For the lunar tides induced on Earth, $\Delta h \approx 53$ cm.

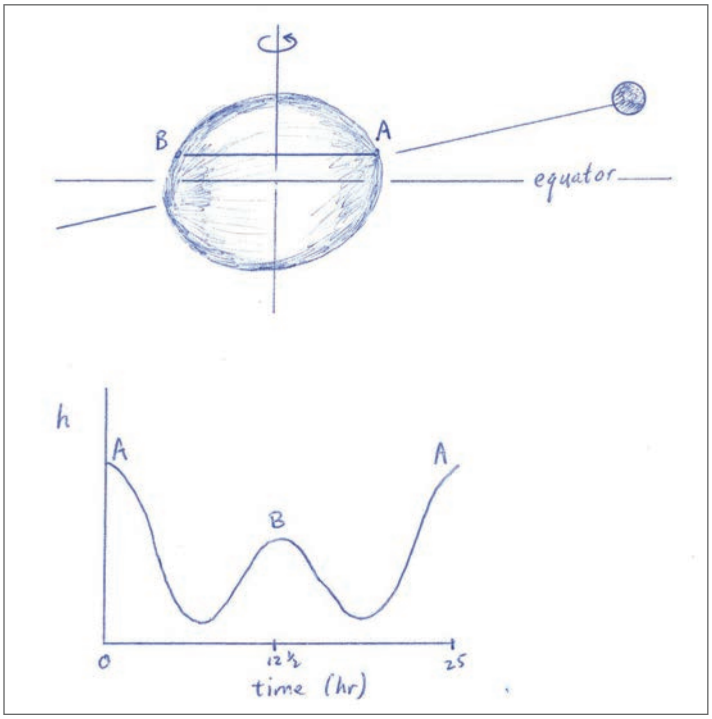

Fig. 4. The diurnal tides. The relative amplitudes of successive high tides depend on the latitudes of the observer and the Moon.

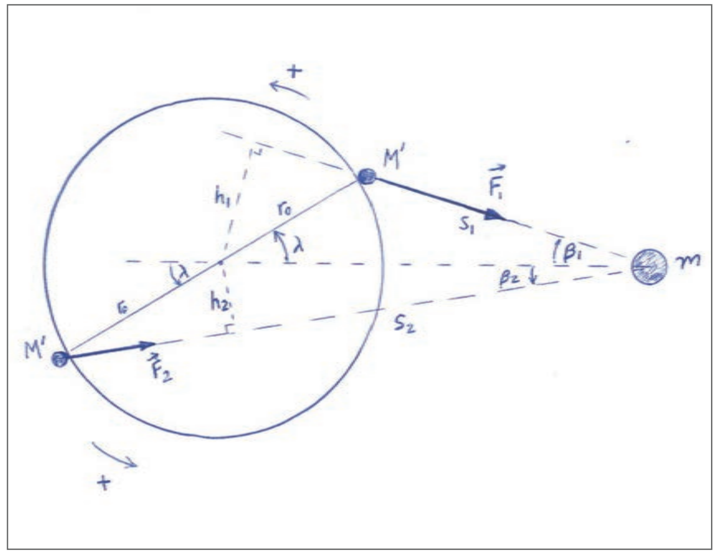

Fig. 5. Model of the Earth–Moon system for studying the torques on the Earth.

The Sun is about 300 million times more massive than the Moon, but it is also about 400 times more distant. Thus, for solar-induced tides on Earth, Δh is about 24 cm, less than half the lunar tide. Solar tidal bulges lie along the line connecting the centers of Earth and Sun. They add to the lunar tide to produce the net tide. (Tides on Earth due to Jupiter and other planets are negligible.) When the Sun–Earth–Moon system is colinear during a new Moon or a full Moon, the solar and lunar tides add constructively to produce maximum high tides, the "spring tides." During the lunar first- or third-quarter phases, the high tides fall to their minimums, the "neap tides."[10]

The Earth's equator tips about 23° relative to the ecliptic, and the Moon's orbit tips 5° from the ecliptic. Tidal bulge peaks can therefore appear at different latitudes within 28° of the equator. As a result, successive high tides at a given location may have different depths. Depending on the latitude, an observer might pass through a bulge's maximum during one high tide but be off the bulge's peak when passing through the opposite one $12\frac{1}{2}$ hours later (Fig. 4).

Interaction of Tides with Land Masses

So far our calculations have conceptualized a simple model that reveals the essentials of tidal dynamics. But the tide–land interactions introduce significant complications. If your boat can make it into the bay only at high tide, a lot rides on having an accurate prediction! When sailing into Half Moon Bay in Auckland, New Zealand, for instance, the equilibrium tidal bulge model must be supplemented with local statistics. Only then can those who live by the tide reliably predict its timing and depth.

As Earth's rotation carries a coastline into a tidal bulge, the tide appears to advance in a relentless wave. In some locations, the shape and dimensions of a bay may introduce tidal resonances with astonishingly large amplitudes. At the Bay of Fundy between New Brunswick and Nova Scotia, high and low tides can differ by some 12 meters! As with musical instruments, such resonances occur whenever the length of the "tube" closed on one end is $\frac{1}{4}$ the wavelength.[11]

Interaction between water and land also has long-term, potentially ominous global effects. Earth's crust rotates about the planet's axis (~400 m/s at the equator) much faster than the speed at which a tidal bulge propagates as a wave.[12] Therefore the rocky bulk of the planet spins beneath the oceans. Since water has a nonzero viscosity, the oceans sliding over the seafloor act like a brake on Earth's crust. This friction pulls on the tidal bulges. Since Earth spins faster than the Moon orbits, the tides lead the Earth-to-Moon axis by a small amount, presently about 10° and denoted λ in the equations to follow. This offset has consequences. The mass M' carried by each bulge gives the Moon's gravitation a "lever arm" by which it torques Earth. Operating in the same direction, tidal friction and lunar gravitational torques slowly decrease Earth's spin.

By Newton's third law, Earth's tidal bulges also exert forces on the Moon. Each of those forces has a component tangent to the lunar orbit that exerts a torque on the Moon, changing its angular momentum and making the Moon recede from Earth. George Darwin articulated this effect in 1897.[13] Let us model it.

Frictional and Lunar Torques on Earth

Here we will lay out the implications of torques on Earth produced by (a) tidal friction as the rotating Earth crust slides beneath its ocean load and (b) the Moon's gravity acting on Earth's tidal bulges. For now we neglect the Sun's torque on the Earth.

The spinning Earth tugs at the water above the ocean floor, while the bulge wants to stay pointed at the Moon. Thus the magnitude of the tidal friction depends on the difference in the angular velocities of Earth's spin and the lunar orbit. Let $\dot{\varepsilon}$ denote the angular velocity of Earth's spin ($\dot{\varepsilon} \approx 2\pi/24$ hr presently), and let the Moon's orbital angular velocity above Earth be denoted $\dot{\theta}$ ($\approx 2\pi/27$ days). The tidal frictional torque will be a function of $(\dot{\varepsilon} - \dot{\theta})$. To first order we may write the frictional torque as $-b(\dot{\varepsilon} - \dot{\theta})$, where b is some constant.

To represent the Moon's gravitational torque on Earth's tidal bulges, we can model Earth as a sphere of radius r_o and treat the bulges as two point masses, each of mass M', lying on the planet's surface along a line inclined at angle λ from the Earth–Moon axis (Fig. 5).

The rotational version of Newton's second law says the net torque acting on Earth equals the rate of change of the planet's angular momentum. When the system is viewed from above

Earth's northern pole, with counterclockwise rotations assigned the positive sign, Newton's second law in terms of torque gives

$$-b(\dot{\varepsilon} - \dot{\theta}) + (F_2 h_2 - F_1 h_1) = I\ddot{\varepsilon}\ , \tag{9}$$

where the double overdot denotes the second derivative with respect to time, I represents Earth's moment of inertia about its spin axis, and the forces F_1 and F_2 are the lunar gravitational forces on the near and far bulge, respectively, with lever arms of lengths h_1 and h_2. From Fig. 5 the force magnitudes are $F_1 = GM'm/s_1^2$ and $F_2 = GM'm/s_2^2$. A lever arm distance can be written $h_1 = r\ \sin\beta_1$, and with the law of sines and small angles, $\sin\beta_1 \approx \beta_1 \approx r_o\lambda/s_1$; similar expressions hold for h_2. To a good approximation $s_1 \approx r - r_o$ and $s_2 \approx r + r_o$. Equation (9) then becomes approximation,

$$-b(\dot{\varepsilon} - \dot{\theta}) + (GmM')(r_o r\lambda)\left[\frac{1}{(r+r_o)^3} - \frac{1}{(r-r_o)^3}\right] = I\ddot{\varepsilon}\ , \tag{10}$$

giving a clockwise torque; hence the angular acceleration is negative and $\dot{\varepsilon}$ decreases with time. Earth's day is presently shrinking by about 0.0023 seconds every century, about 2 hours every 300 million years. Assuming this rate to be constant (which it isn't, but the assumption nonetheless offers an order-of-magnitude long-term estimate), 900 million years ago the day would have been about 18 hours long, a conclusion consistent with fossil records from that era.[14]

At this deceleration rate, 3 billion years ago the day would have been about 6 hours long! Prevailing winds of 100 mph or more would have been normal. With the smaller solar-induced tides and violent wind-driven wave action on shores, the transition of life from the sea to the land would have faced extra challenges! Had the Moon never come into existence, life on this planet would be very different.[2] We owe the Moon some profound appreciation. We may live with it and not by it, but our physiology and routines of life have adapted to its rhythms.

As a player in tidal friction, the tidal bulge offset angle λ depends on the difference between Earth's spin frequency and the Moon's orbital frequency. If Earth's spin period were the same as the lunar orbital period (with the Moon in geosynchronous orbit), the Earth–Moon system would rotate as if it were a rigid body, with the tidal bulges aligned along the Earth–Moon axis. Should the Moon somehow orbit faster than Earth spins ($\dot{\varepsilon} < \dot{\theta}$), the tidal bulges would lag the Earth–Moon axis, changing the sign of λ. The Moon would appear from Earth to rise in the west and set in the east. Evidently, λ is a function of ($\dot{\varepsilon} - \dot{\theta}$).

Earth's spin will continue to slow down, with λ decreasing all the while, until $\dot{\varepsilon} \to \dot{\theta}$ and $\lambda \to 0$. When $\dot{\varepsilon} = \dot{\theta}$ one of the Earth's hemispheres will face the Moon all the time, an effect called "tidal phase locking." Then $\lambda = 0$ and the lunar torque exerted on Earth will vanish.

Now let's switch perspectives and think about the tidal forces exerted by Earth on the Moon. The Moon is already phase locked to Earth; the satellite's spin period equals its orbital period, so the Moon always presents the same face to us. It was not always this way. Early in the history of the Earth–Moon system, a Moon newly formed by accretion would have been hot and relatively fluid. Using Eq. (8), the max-to-min tides on the Moon (due to Earth) compared to those on Earth (due to the Moon) would have been

$$\frac{(\Delta h)_{Earth\ on\ Moon}}{(\Delta h)_{Moon\ on\ Earth}} = \left(\frac{M}{m}\right)^2 \left(\frac{\rho}{r_o}\right)^4, \tag{11}$$

where ρ denotes the lunar radius. Because $M \approx 100m$ and $\rho \approx \frac{1}{4}r_o$, Earth would produce tidal bulges about 40 times larger on the Moon than those the Moon produces on Earth. Thanks to large tidal torques [15] and the small lunar moment of inertia, the Moon phase locked to Earth early in its history. Thus the Moon keeps only one face toward Earth today, with its spin and orbital periods equal. The now-frozen lunar tidal bulges, aligned along the Earth–Moon axis, offer no lever arm for Earth to exert torque on the Moon.

However, Earth's tidal bulges are not yet aligned along that axis, so they exert a torque on the Moon relative to Earth's center. To analyze those torques, let's reuse Fig. 5, invoke Newton's third law, and evaluate the torques with respect to Earth's center. The net torque changes the lunar angular momentum, which can be partitioned into the orbital angular momentum *of* the Moon's center of mass revolving about Earth, plus the lunar spin angular momentum *about* the Moon's center of mass.[16] Newton's second law, written in terms of torque, now says

$$F_1 h_1 - F_2 h_2 = \frac{d}{dt}(mr^2\dot{\theta} + i\dot{\theta}), \tag{12}$$

where we have invoked the equality of the Moon's orbital and spin angular velocities, and i denotes the lunar moment of inertia about its spin axis. The torque computations are similar to our previous case but with a crucial change of sign, leaving

$$(GM'm)(r_o r\lambda)\left[\frac{1}{(r-r_o)^3} - \frac{1}{(r+r_o)^3}\right] = \frac{d}{dt}(mr^2\dot{\theta} + i\dot{\theta}). \tag{13}$$

This time the term in square brackets is positive, which implies

$$2mr\dot{r}\dot{\theta} + (mr^2 + i)\ddot{\theta} > 0. \tag{14}$$

The Moon's orbit about Earth is well described by Kepler's third law, which says that $\dot{\theta}^2 \sim 1/r^3$. Differentiating this with respect to time means that $2\dot{\theta}\ddot{\theta} \sim -3\dot{r}/r^4$. Using Kepler's third law again to substitute r for $\dot{\theta}$, we find that $\ddot{\theta} \sim -1/r^{5/2}$. For large r and short times (astronomically speaking), the lunar angular acceleration is very small and the orbital angular velocity approximately constant. Thus it seems safe to say that $2mr\dot{r}\dot{\theta} > 0$, which means that, in our era, $\dot{r}$ cannot be zero or negative. Thus $\dot{r} > 0$, which means the Moon recedes from Earth.

Only after the Apollo 12 astronauts left reflectors on the Moon capable of reversing the direction of a laser beam could the value of $\dot{r}$ be measured. It turns out to be about 3.8 cm/yr.[17]

Long-Term Future

For the foreseeable future the Earth will keep rotating as the Moon revolves around it. As the Moon recedes, its orbital period will slowly lengthen, and Earth's spin will gradually slow down. Those effects will continue, with the phase-locking mechanism making continual adjustments, until Earth's rotation period matches the Moon's orbital period. The Moon and Earth will then be *mutually* phased-locked to each other. The Moon will not be visible from half of Earth's surface! The tidal bulges will still be there, but they will lie along the Earth–Moon axis. At a fixed latitude, high and low tides will no longer occur. Earth's day will be about 50 times longer than it is today, and the Moon's orbital radius will be about 1.4 times its present value.[18] For those on Earth who can see it, the Moon will subtend an angle about 70 percent of its present angular diameter in the sky. Solar eclipses will be annular, not total.

However, the Sun will continue exerting tidal forces on Earth, causing tidal stresses, tidal bulges, and tidal friction, though diminished from the lunar-induced case. With the solar-induced tidal friction operating, Earth's spin will continue to slow after the Moon and Earth become mutually phase locked. If the Moon could be dismissed, Earth and the Sun would eventually become phase locked. With the Moon still in action, the solar-induced torque will slow Earth's spin *below* the value at which the planet phase locked to the Moon. Then the Earth will spin more slowly than the Moon orbits, and the Moon will appear from Earth to rise in the west and set in the east. Lunar tides will reappear with torques opposite from those of our era. Those future lunar-induced tidal bulges will lag instead of lead the Earth–Moon axis. Instead of receding, the Moon will approach the Earth. The closer the satellite gets, the stronger the tides, until the Moon comes within the Roche limit.

To estimate the upper bound for the orbital radius at which tidal forces break the Moon apart, neglect the chemical bonds that hold the rocks together and imagine the Moon is held together only by its self-gravitation.[19] Return to Eq. (4) after interchanging the roles of Earth and the Moon, and look at the stress along the Moon's z axis. When the lunar self-gravitation becomes less than the z component of the tidal force, the Moon will disintegrate; that is, when

$$\frac{Gm}{\rho^2} < \frac{2GM\rho}{r^3} \qquad (15)$$

or $r < (2M/m)^{1/3}\,\rho$. Since $M \approx 100m$ and $\rho \approx 1.74\times10^6$ m, the Moon would come apart when $r < 10^7$ m ~ $10r_o$. Perhaps Earth will then have a Saturnian ring system! The actual Roche limit is smaller than this crude estimate because the Moon's rocks are held together by internal forces. The Moon presently orbits at about $60r_o$, well outside our pessimistic Roche limit.

All of this would take billions of years, much longer than the lifetime of the Sun.[20] Four to five billion years from now, the Sun will deplete the hydrogen in its core. Nuclear collisions fusing hydrogen to helium today will be quenched. Gravity will squeeze the solar core tighter, raising its temperature. Before the core temperature hits the 100 MK or so necessary to fuse helium to carbon, the layers around the core will reach the 15 MK or so necessary to start hydrogen fusing to helium. Shells of fusion working their way through the star will swell it into a red giant that will engulf the Earth–Moon system.

However, this planet, moonlight walks, sunrises and sunsets, and the starry sky are no less beautiful for their being temporary. Get out there and enjoy them! Appreciate what we have, and be aware of what it means. We are part of nature, not a spectator on the sidelines. Connect with the many worlds around us!

Acknowledgments

I am grateful to Thomas Olsen and Mark Winslow for reviewing a draft version of this article.

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[1] Statement by Charles Kuralt in the CBS Documentary "The Moon Above, The Earth Below" (1989), broadcast on the 20th anniversary of the flight of Apollo 11.

[2] Neil F. Comins and William J. Kaufmann III, *Discovering the Universe*, 5th ed. (W.H. Freeman, New York, NY, 2000). See the provocative essay on p. 146, "What if the Moon Didn't Exist?" See also Comins' essay by the same title on a website of the Astronomical Society of the Pacific, http://www.astrosociety.org/education/publications/tnl/33/33.html.

[3] Michael Seeds and Dana Backman, *Horizons: Exploring the Universe*, 12th ed. (Brooks-Cole, Boston, MA, 2012), p. 68.

[4] Comins and Kaufmann (Ref. 2), p. 140.

[5] Because fields from a point source go as inverse distance squared, the point-sphere equivalence theorem for gravitation (Newton's law of universal gravitation) and electrostatics (Coulomb's law) holds. This can be shown with integration or by Gauss' law. See any calculus-based introductory physics text with chapters on gravity, Coulomb's law, and Gauss' law.

[6] See John C. Taylor, *Classical Mechanics* (University Science Books, Sausalito, CA, 2005), p. 296; Jerry B. Marion and Stephen T. Thornton, *Classical Dynamics of Particles and Systems*, 5th ed. (Thornson Brooks/Cole, Belmont, CA, 2004), pp. 287–289.

[7] See Hans C. Ohanian, *Gravitation and Spacetime* (W.W. Norton, New York, 1976), pp. 26–32; Taylor (Ref. 6), pp. 330–336; Marion and Thornton (Ref. 6), pp. 198–204.

[8] Another noninertial acceleration comes from Earth's spin, which carries angular velocity $\omega = 2\pi$ rad/day $= 7.3\times10^{-5}$ rad/s. A point on Earth's equator gets carried around the spin axis with the acceleration $r_o\omega^2 \approx 4\times10^{-8}$ m/s^2 (~25 times smaller than tidal forces), directed toward the local vertical. See "When '**F**' does not Equal 'm**a**,'" *SPS Observer* (Spring/Summer 2001), pp. 10–14.

[9] The word "diurnal" may seem less strange if we compare it to the more familiar "nocturnal."

[10] The "spring" tide's name is not seasonal but comes from the notion of the tide "springing up" higher than usual. According to *Webster's Dictionary*, "neap" comes from the Middle English *neep* or the Anglo-Saxon *nep*, which meant "scarcely touching."

[11] One might ask how a pulse of water, such as an incoming tide, can have "a wavelength." A wave pulse can be built by a superposition of harmonic waves.

[12] Left to itself, the tidal bulge, as a heap of water, would form a wave pulse, a superposition of waves. In deep water the phase velocity of a surface wave is $\sqrt{(g\lambda/2\pi)}$ for wavelength λ. Everyday waves have wavelengths on the order of 100 m or less, giving speeds on the order of 10 m/s or less. In contrast, tsunamis can travel over 500 mph, with wavelengths on the order of 100 km. See, e.g., http://walrus.wr.usgs.gov/tsunami/basics.html.

[13] George Howard Darwin, *The Tides and Kindred Phenomena in the Solar System* (Houghton, Mifflin and Co., Boston, MA, 1898), Chs. XVI and XVII, "Tidal Friction" and "Tidal Friction (continued)." This book was adapted from Darwin's Lowell Lectures at the Lowell Observatory in Boston in 1897. George Darwin (1845–1912) was a highly decorated astronomer, professor at Cambridge University, President of the Royal Society, and (in case you are wondering) the son of Charles Darwin.

[14] Seeds and Backman (Ref. 3), p. 70. 900 million years ago the only life on Earth was algae and bacteria, and they were sparsely distributed; see, e.g., Bill Frio, *Geology of the Great Basin* (University of Nevada-Reno Press, Reno, NV, 1986), p. 32.

[15] The strong tidal torquing of the early Moon would keep its interior hotter for a time longer than otherwise; a similar thermodynamics seems to be operating on Jupiter's moon Europa today, which has a deep-water ocean beneath its frozen crust, despite the intense cold of the Jovian region. Radioactivity also provides a long-lasting heat engine for planetary structures.

[16] See any introductory physics textbook, or "Angular Momentum and 'Spin,'" *SPS Observer* (Fall 2000), pp. 10–14. The theorem follows because angular momentum is additive.

[17] It is an elementary exercise in geometrical optics to design a set of mirrors that will return a light ray back along its incident direction. The arrays of reflectors built into automotive taillights use the same geometry.

[18] Robert B. Gordon, *Physics of the Earth* (Holt, Rinehart, and Winston, New York, 1972), p. 69.

[19] Bradley W. Carroll and Dale A. Astlie, *An Introduction to Modern Astrophysics* (Addison-Wesley, Reading, MA, 1996), pp. 766–767.

[20] Longer than the Sun's lifetime follows from a back-of-the-envelope estimate. For the Earth to become phase locked to the Moon, assuming nothing else changes, the Earth would have to lose about 26 hours. If the spin slow-down rate would stay at 0.0023 s/century for all time, this would take only 280 billion years! Of course, as the Moon recedes, its orbital period changes, so the coupled situation gets rather dynamic. But the point is, for large bodies the phase-locking timescale may be comparable to a star's lifetime. However, many planetary satellites besides our Moon are already phase locked to their planets. Pluto and its satellite Charon are already mutually phase locked.