An Idea for introducing Students to Statistical Mechanics:

Dice Landing Probabilities

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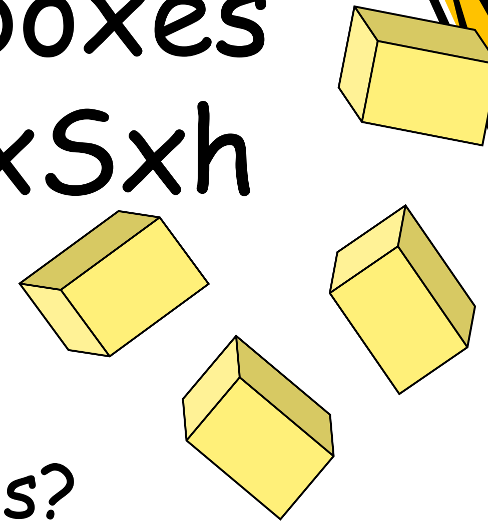
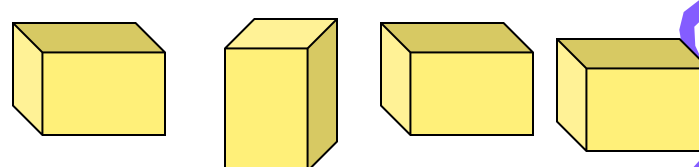
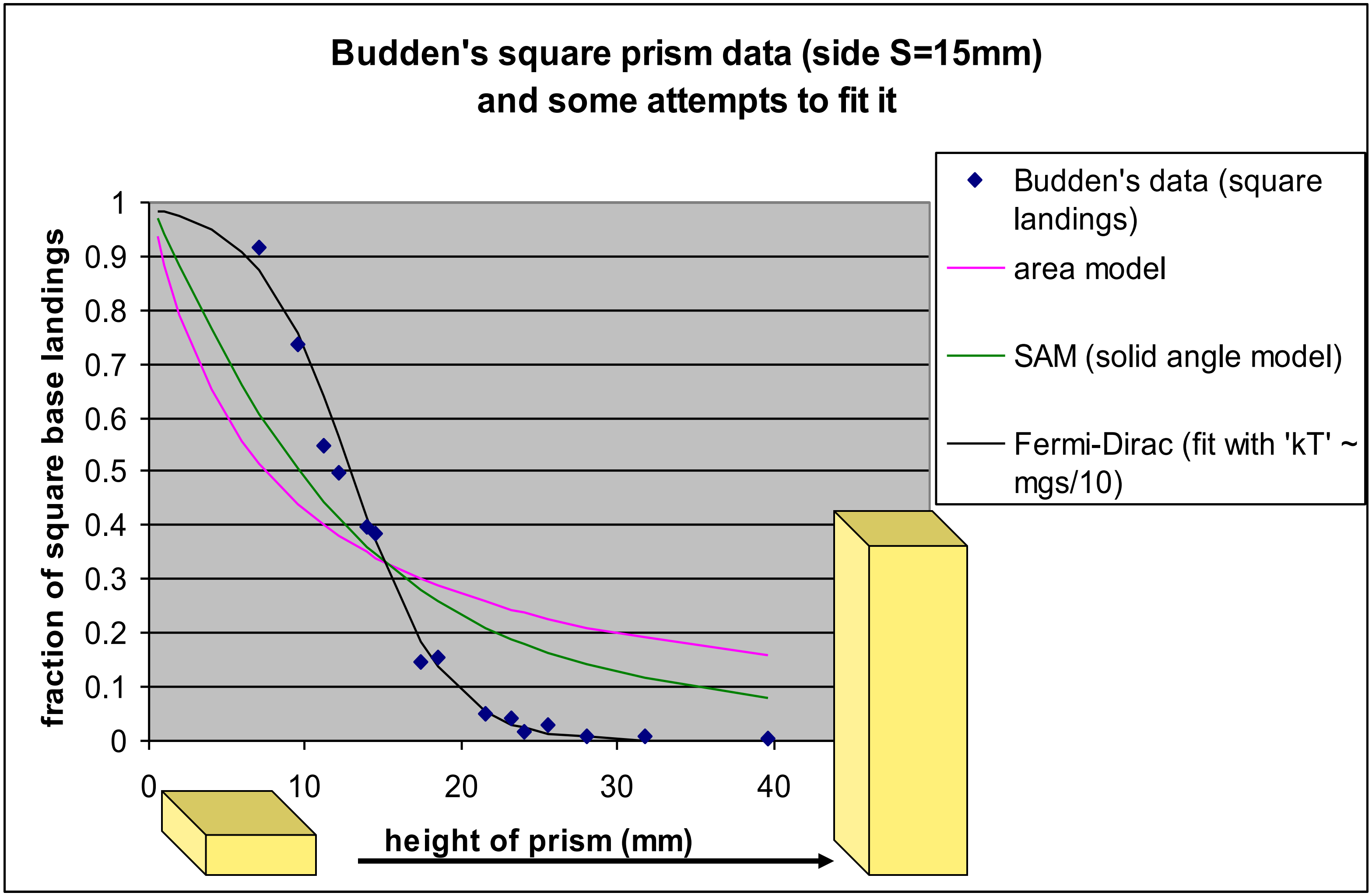
If you drop some boxes with dimensions SxSxh



Which do you think will be a better predictor for the fraction of square base landings?

a) Area of base square as a fraction of total surface area
b) Fractional solid angle subtended by base square from center of box
c) The Fermi-Dirac distribution

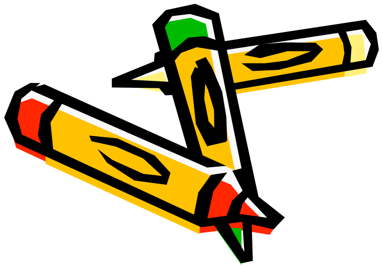




Surprised? I was…

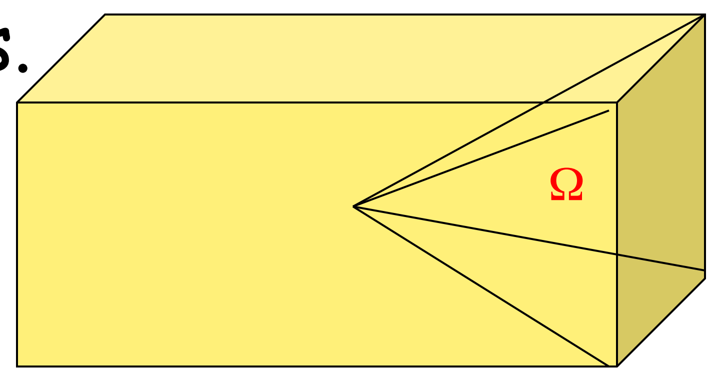
Why study droppings?

- Math exercises for grammar school (F. Budden, D Singmaster in the *Mathematical Gazette*, Heilbronner, Berkshire, non-cube box dice, 1980-85)
- Gambling and loaded dice (E. Levin in AJP, brass slugs in plastic dice, 1983)
- Coins landing on edge (H. Bondi in EJP, D. Murray and S. Teare, in PhysRevE, 1993, and in Murray's thesis, 1991, building on work of Yue, Zhang, Keller, Vulovic, Prange, Feldberg and others)
- Complete list of fair dice (E. Pegg, UC at Colorado Springs master's thesis, 1997)
- And probability experiments, undergraduate research, Mars landings (GW, C. Gresham, D. Lutterman and other students, starting in 1992, still unpublished)



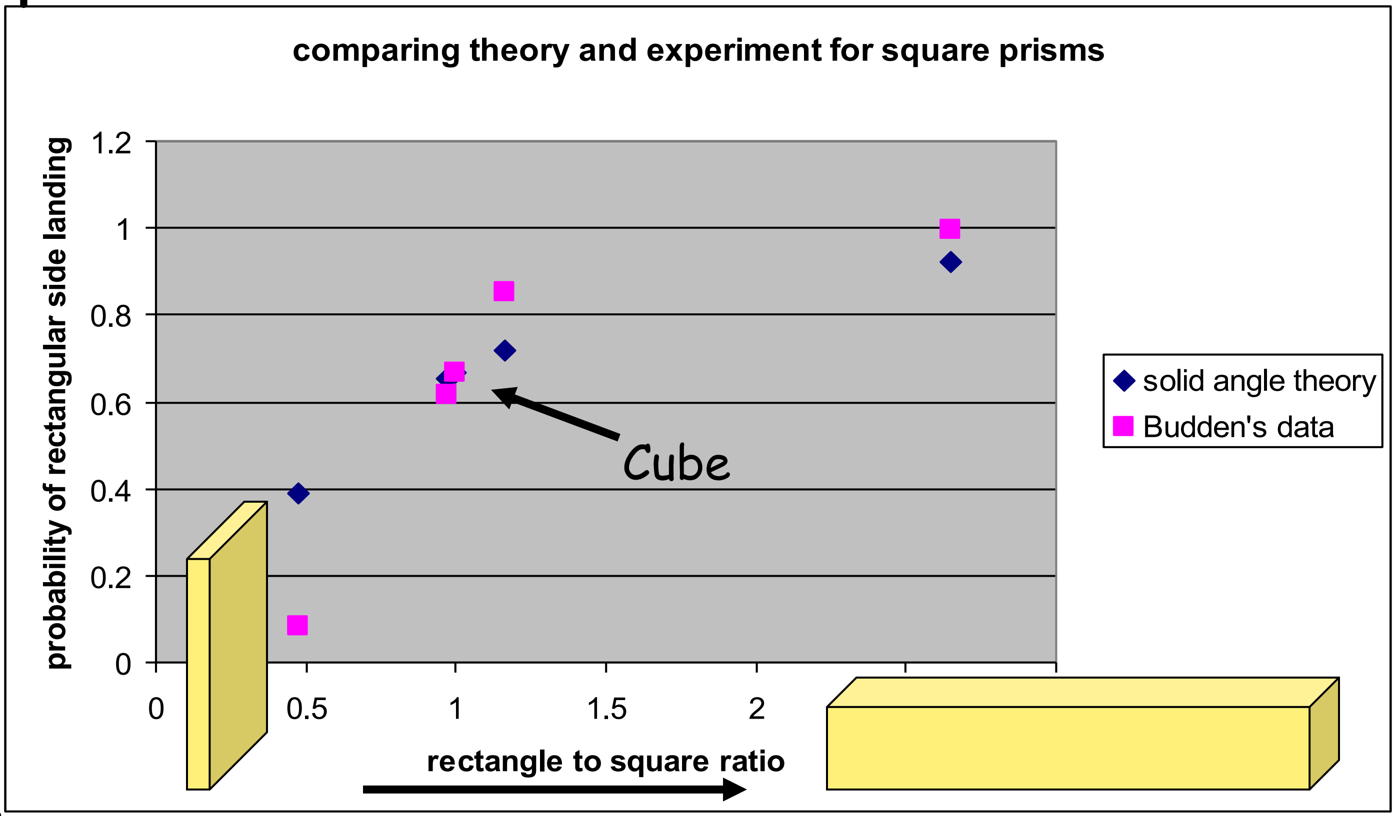
A theoretical idea

- David Singmaster (1981) appears to have been the first to suggest the solid angle model---the solid angle from the center is proportional to the number of landings on that side. For boxes, this is exactly what you'd expect if the box were balancing on a corner and compelled to rest on the side below the center of mass.



Solid angle theory gets trend right but Singmaster was disturbed that the theory didn’t do better for squatty and tall prisms:

…”I am perplexed that the theoretic results are so far from the observed results and I can see no geometric explanation.”



Singmaster’s model predicts 8% edge landings when applied to a “10p” coin. The editor of *The Mathematical Gazette* asks how to adapt the theory to make it a better model.

What about friction, restitution, etc?

- H. Bondi (of cosmology fame) addresses the issue of a cylinder dropping in 1993 from a purely theoretical mechanics perspective. He starts with an inelastic, perfectly rough floor, then a smooth floor…he finds probability of side landing for coin depends dramatically on the height from which it is dropped among other things. For nickels his model gives about 0.6% edge landings when dropped from 10cm or so.

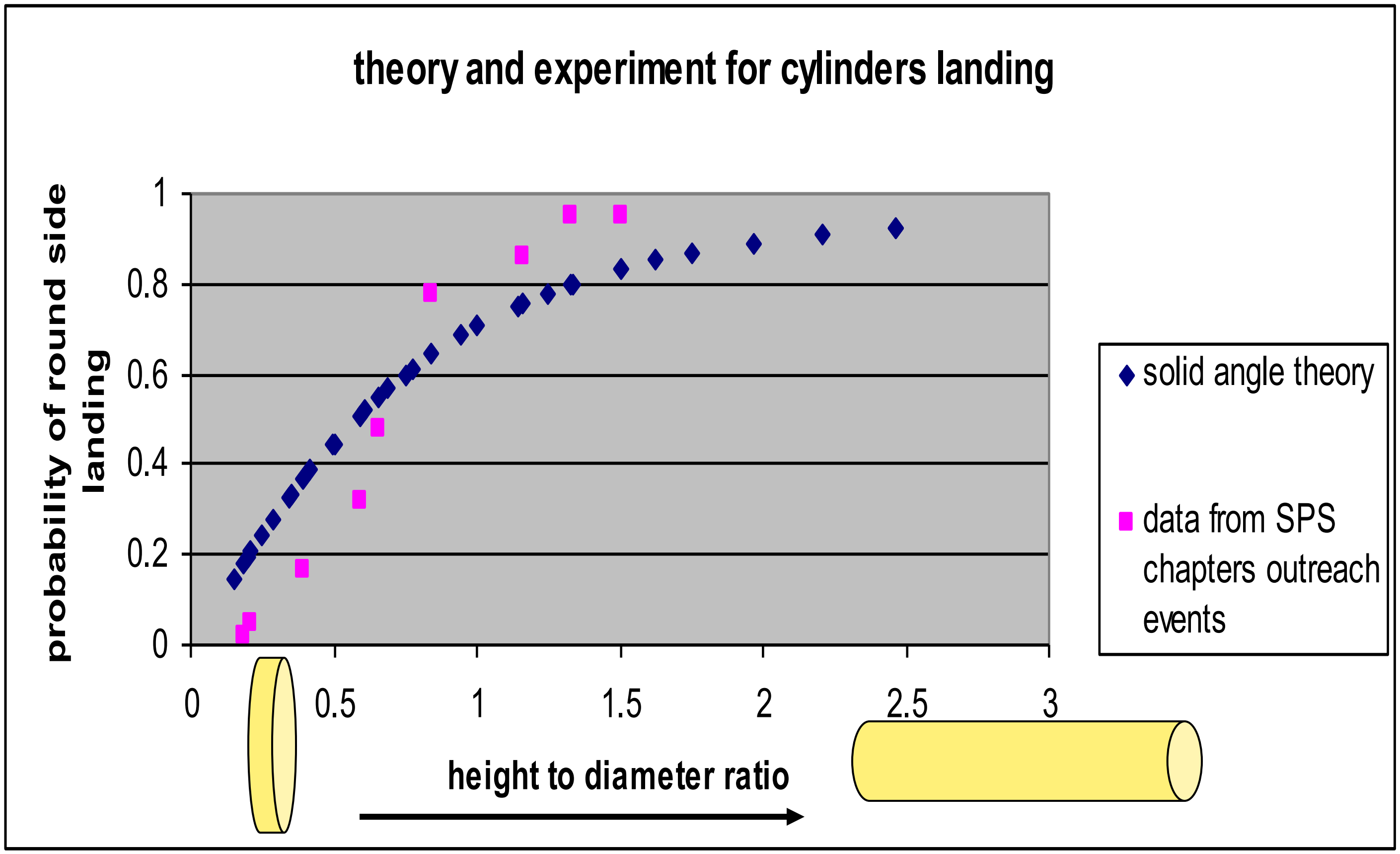
What about experiments?

- Daniel Murray and Scott Teare, also in 1993, simulate the bouncing of a dropped cylinder on a frictionless surface allowing various coefficients of restitution, using experiments with hex nuts to help determine appropriate values for parameters in their model. They predict a nickel will land on its edge 0.017% when dropped from about 15cm.

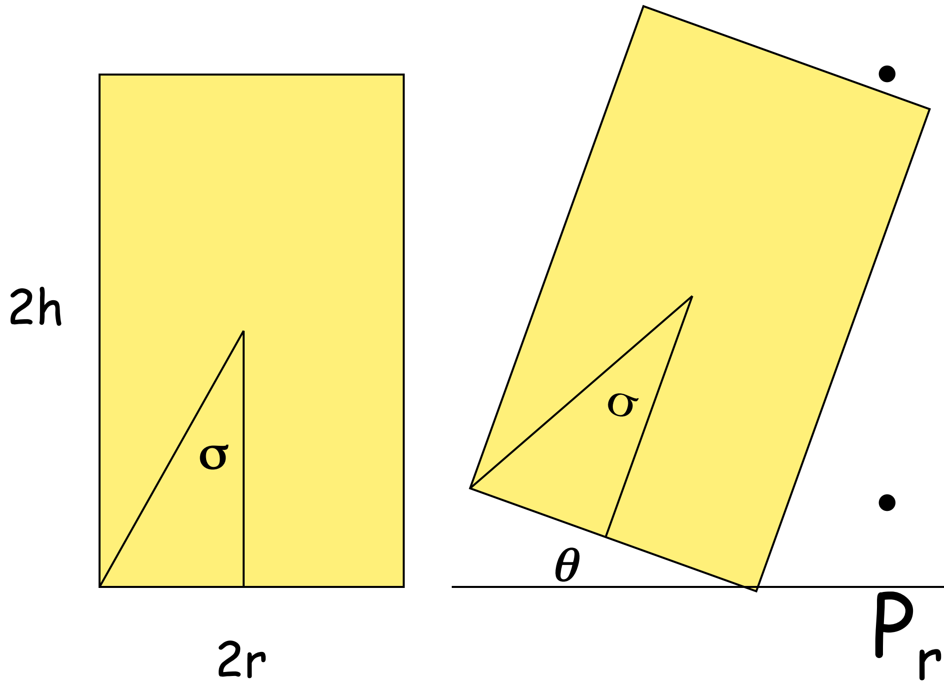
How does the solid angle theory do for cylinders?

Not bad overall…

But notice that the solid angle theory predicts too many rolling landings for coins and too few rolling landings for pillars reproducing Singmaster's lament



A modified solid angle model for a rectangle in 2-D



Tan $\sigma = r/h$

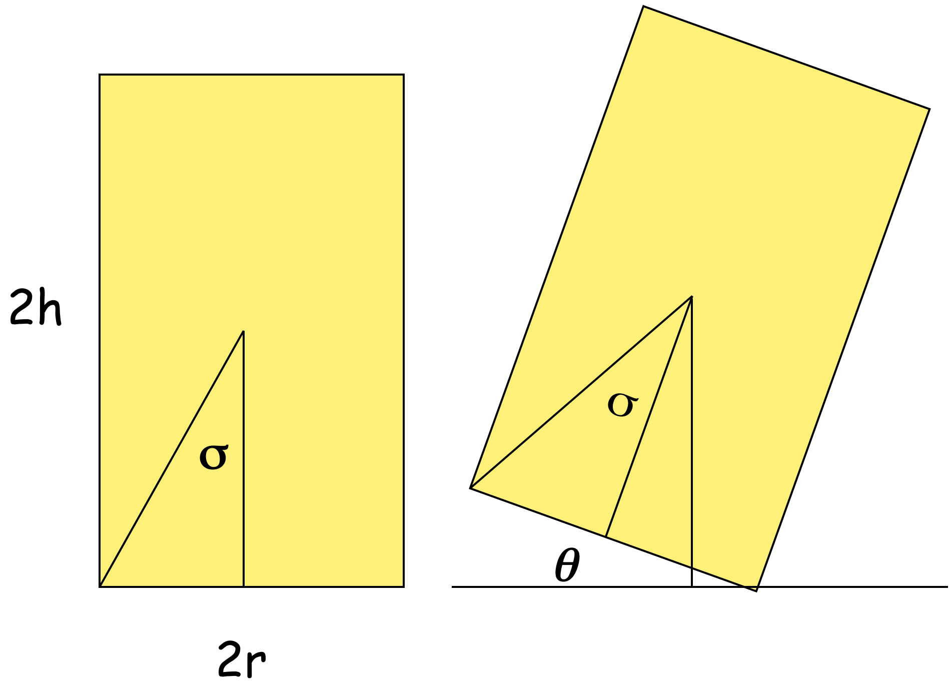
- Solid angle theory gives probability of r-side landing as $P_r = 4\sigma/2\pi$, or
- $P_r = \int_0^{\sigma} d\theta \Big/ \int_0^{\pi/2} d\theta$

P_r = Arctan(r/h)/(π/2)

…but in an ensemble of settling rectangles, the orientations of the rectangle which have the c.m. higher are less likely to be present than those with lower c.m. How much less likely?

Let's try a Boltzmann factor of exp(-constant*E) or since KE is very low, let's weight each possible orientation with a factor of $\exp(-\beta y_{cm})$

A modified solid angle model for a rectangle in 2-D



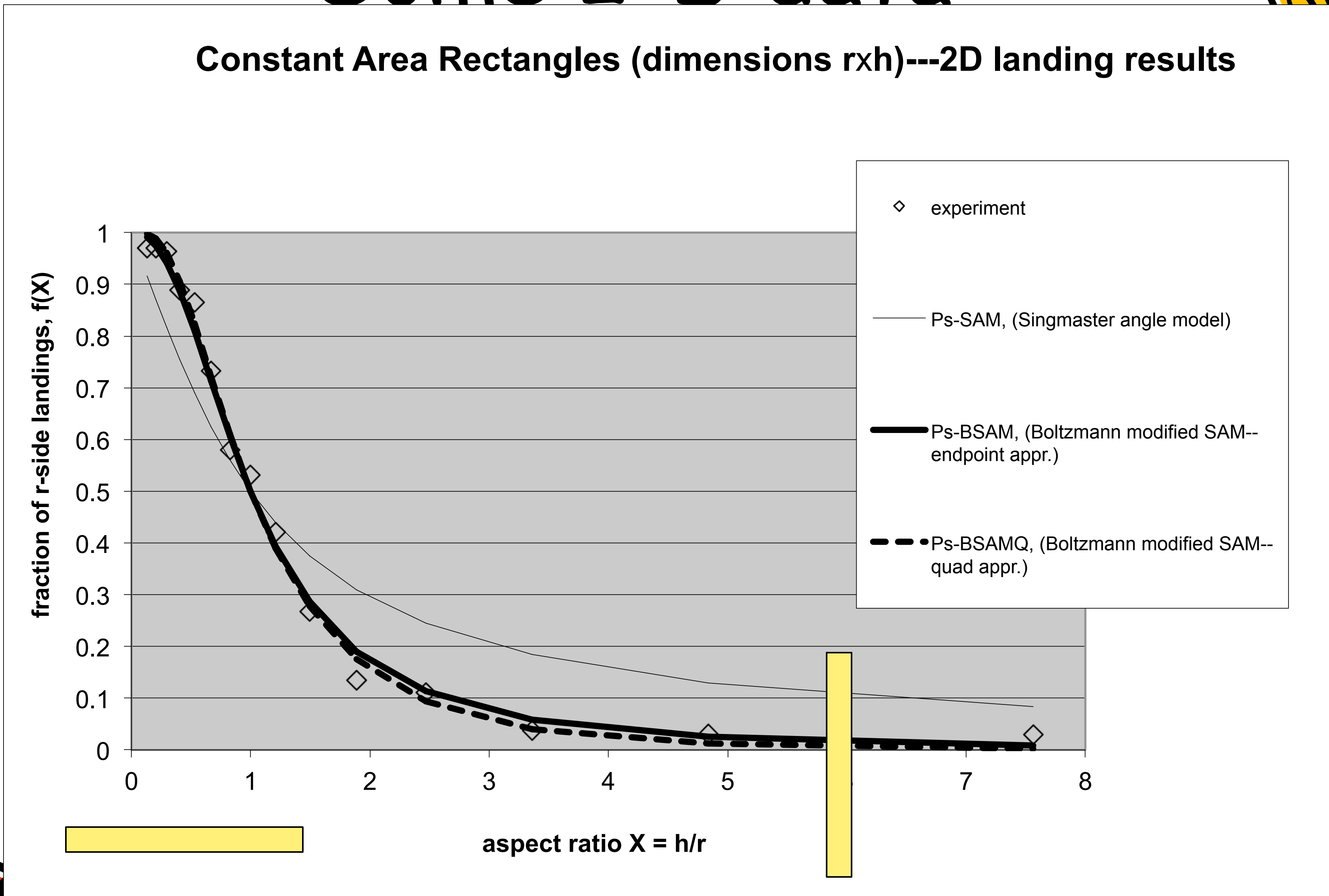
So we get

$$P_r = \int_0^{\sigma} e^{-\beta y_{cm}} d\theta \Big/ \int_0^{\pi/2} e^{-\beta y_{cm}} d\theta$$

$y_{cm} = [r*\sin(\theta) + h*\cos(\theta)]/L_{ch}$

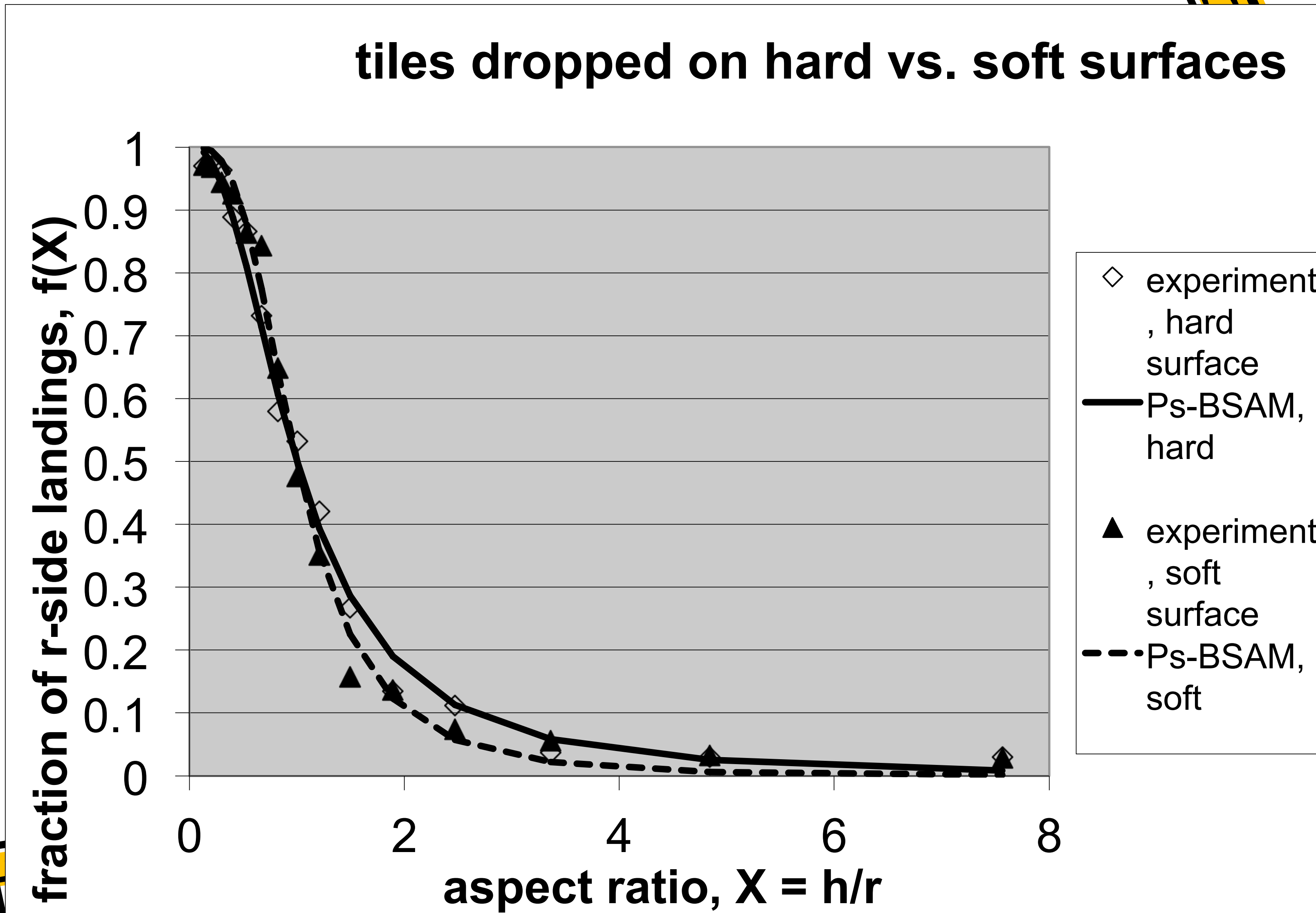
Notice that this modification will reduce the number of high c.m. landings, and will increase the number of low c.m. landings

Some 2-D data



Constant Area Rectangles (dimensions rxh)---2D landing results
fraction of r-side landings, f(X)
aspect ratio X = h/r
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5
7
8
experiment
Ps-SAM, (Singmaster angle model)
Ps-BSAM, (Boltzmann modified SAM--endpoint appr.)
Ps-BSAMQ, (Boltzmann modified SAM--quad appr.)

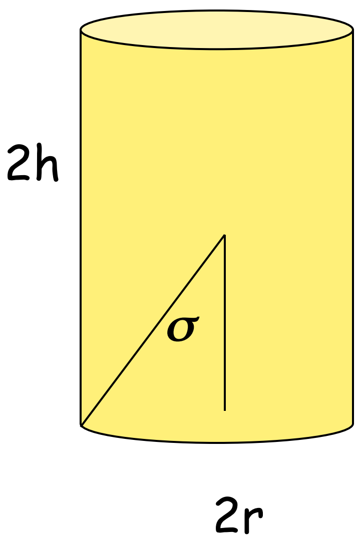
More 2-D data



tiles dropped on hard vs. soft surfaces
fraction of r-side landings, f(X)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
aspect ratio, X = h/r
experiment, hard surface
Ps-BSAM, hard
experiment, soft surface
Ps-BSAM, soft

Move this to 3-D

- $P_h = \int_0^{\pi/2} d\phi \int_0^{\sigma} e^{-\beta y_{cm}} \sin(\theta) d\theta \Big/ \int_0^{\pi/2} d\phi \int_0^{\pi/2} e^{-\beta y_{cm}} \sin(\theta) d\theta$

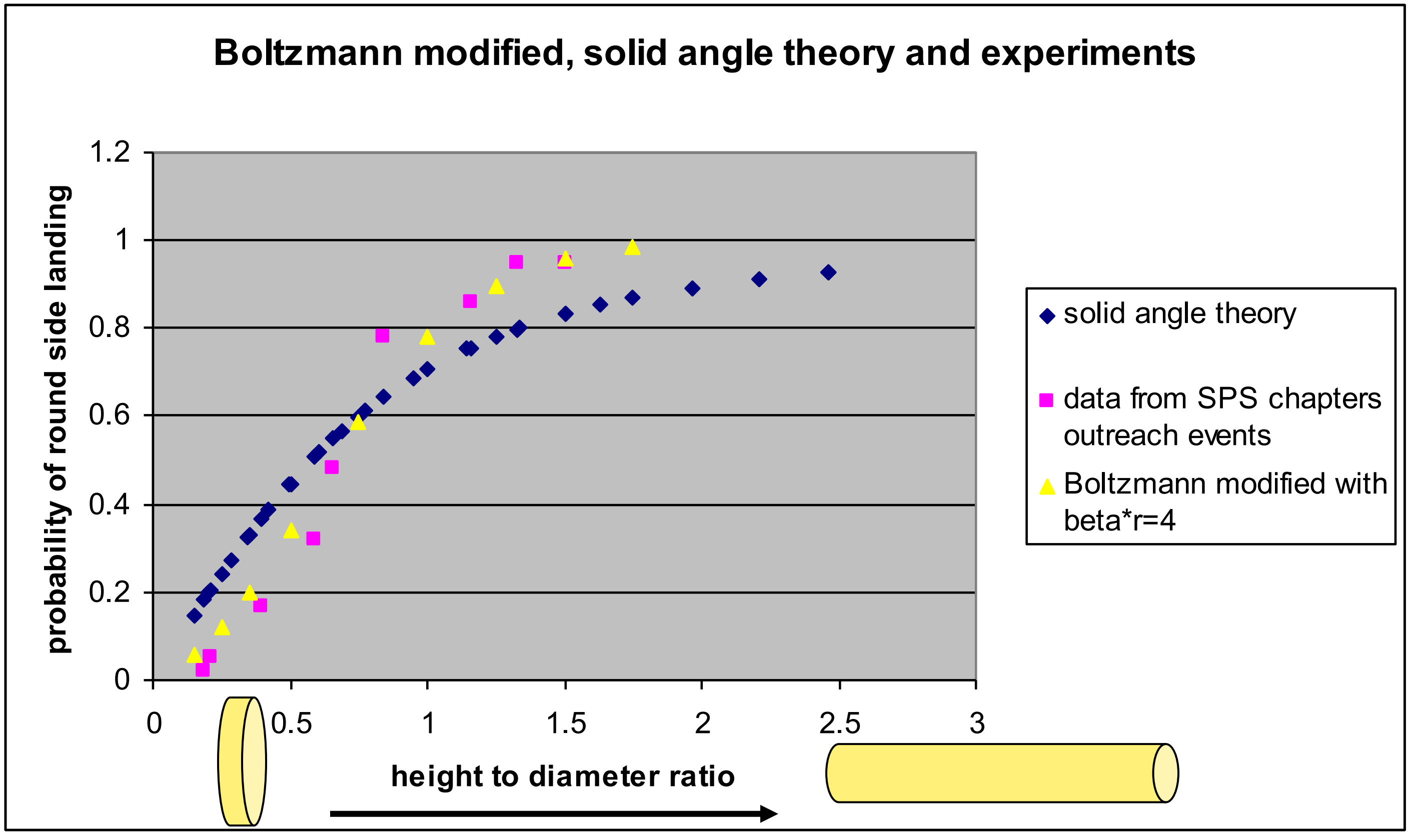


with

$y_{cm} = r*\sin(\theta) + h*\cos(\theta)/L$

$\tan \sigma = r/h$

Encouraging



My prediction for a nickel landing on its edge?

- Hmmm…on a hard surface or soft?
- From 15cm or 2 meters?
- Cylinders, or actual nickels?

- I’m pretty sure it’s between 0% and 8%

Summary

- I believe dropping experiments are good ways to introduce probability and distributions.
- The SAM-Boltzmann model seems to describe a variety of dropping data pretty well.

Thanks to

SPS chapters at

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And you!

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